ELEMENTARY ALGEBRA

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PREFACE

The authors make no apology for offering another algebra to the school public. In influential places algebra has been challenged as a suitable subject for high school pupils. Is it not the part of wisdom, before eliminating a subject of so long and undisputed standing as algebra, to try reconstructing and improving its form and even some of its substance? The authors believe that this text has accomplished much in both of these particulars.

This book is not written, however, with the thought of defending an unworthy claimant to a place in the curriculum. The true view is that the high educational merit of school algebra may be raised even higher by a treatment whose language and mode of exposition are in accord with the possibilities and appreciations of youth, and whose scientific soundness is at the same time not seriously compromised. It is the authors' conviction that rightly taught, algebra is of great educational value, and that to most high school students it is not distasteful.

In carrying out their views on this line, the authors have attempted several specific things. Some of these stated briefly are as follows:

1. To present the material in a language and mode that are simple and at the same time mathematically sound, without resort to mathematical technicalities.

2. To motivate the various topics of algebra either through special problematic situations, or through the gradually rising demands of the equation for particular phases of algebraic technique. As examples see pages 27, 32, 59, 266, etc.

3. Persistently to make the first steps into the treatments of algebraic subjects through the analogous subjects of arithmetic. (See pages 20, 41, 91, 107, 180, 229, etc.)

4. To give the pupil some really valuable help in learning to read, to comprehend, and to interpret algebraic language, and to express mathematic principles and rules in this language. Chapter XIII on General Numbers, Formulas, and Type-forms may be cited as a good illustration of this treatment.

5. To give an early introduction to simultaneous simple equations and to complete their study by recurrent treatments as the course develops.

6. To make early and frequent use of the graph freed from analytical technicalities, *as an aid to the development of algebra* through clarifying and vivifying meanings of algebraic processes and technique in the *beginning stages* of teaching and learning them.

7. To seek diligently for such an order of treatment of the special topics as is dictated by the highest economy in the mastery of the elements of the science of algebra. By this means it is hoped to give a stronger and a more highly educative first-year course in the customary time. (See Table of Contents.)

8. Carefully to grade as to difficulty and to balance as to quality and quantity the problems and exercises of the book, again with an eye single to the unfolding needs of algebra. (See problem-lists given under the different topics.)

9. To correlate with arithmetic, geometry, general science, and everyday life to as great a degree as the best school interests of first-year algebra require.

10. To heighten the workability of the text by a synoptic table of contents, a summary of definitions (page 322), and a good working index.

A little of the pedagogical background of the organization of this text may be stated here. The authors hold the view that teachers of present-day secondary algebra should recognize that they are under three significant professional obligations to their pupils, *viz.*:

I. *To rationalize the analogous arithmetic of the algebraic topics taught.*

It is hardly reasonable to expect of beginners in secondary algebra that they really *understand* their arithmetic, even as arithmetic. Still less may secondary teachers rightfully expect that beginning pupils have grasped their arithmetic in such form that it can be made the basis for algebra. This is a much more difficult matter because, although both arithmetic and algebra are abstract sciences, algebra involves a much higher order of abstractness than arithmetic.

In view of the scope and complexity of modern elementary school arithmetic, of the slight emphasis of school officials, examiners, and surveyors, and even of school programs upon rationalizing processes, it is worse than useless to expect, let the most conscientious teacher strive as he may, that more be done in the elementary school than to rationalize the most elementary notions and processes of arithmetic. In fact, for several years elementary teachers have been urged by some authorities to renounce rationalizing for mere habituating and drill procedures. These things, coupled with the fact that arithmetic of the sort covered in our grammar grades is one of the most difficult of all mathematical branches, and with limitations of program time and immaturity of pupils, hopelessly preclude any attempts at those far-reaching inductions and generalizations that are essential at the very beginning of rational algebra. Therefore, this fundamental work for the highly specialized needs of the several algebraic topics belongs properly to the algebra teacher. This text supplies the initiatory arithmetical rationalizing for the algebraic topics and subjects at the precise places where it is needed and of the sort that is appropriate.

II. *To show that many algebraic things can be done geometrically, i.e.,* by the aid of the concrete space material of diagrams, pictures,—of any graphical helps to clear thinking.

To see, to calculate, and to comprehend is the true order of steps in mastering algebraic tasks. The concepts of lines, rectilinear figures, and solids are so much space material, always and everywhere available for concreting, visualizing, and vivifying number laws and relations, at no great cost in money or effort. The high school youth has lived long enough in this world of space to have become familiar with it, and his spatial experiences need only to be drawn upon to enable him to lay firm hold on the highly abstract fundamentals of beginning algebra. Really to see that algebra merely generalizes mensuration laws, that algebraic numbers, laws, and problems picture into vivid forms, and to learn the secret of laying before his eyes diagrammatically the conditions of algebraic problems as an aid in formulating these conditions into algebraic language and technique, are of the highest interest and value to the beginner. The professional duty of employing the concreting agencies of pictures, diagrams, geometrical figures, and graphs to vivify and vitalize algebra will be readily accepted by the teacher who strives to realize in practice the educational merits of well-taught algebra. No clumsy laboratory equipment of extensive and expensive apparatus is required to enable the algebra teacher through space-materials to supply genetic backgrounds for algebraic problems, truths, and laws.

III. *To show the pupil that algebra will enable him to do much more than he can do with either arithmetic or geometry, or both.*

The first and second professional duties are really preliminary, through which motivating and clearing the way for effective attack are accomplished. This third duty is peculiarly due to algebra. It is in fact due to both pupil and subject that the particular gains to be secured by a mastery of the subject-matter shall appear in the learning acts.

For example, the pupil should see such things as, that by arithmetic he cannot subtract if the subtrahend happens to be greater than the minuend; that he cannot solve so simple an equation as $x+9=3$; but that if he include the negative numbers among his number notions he can do both easily. He should see that he can square and cube numbers geometrically, but that he can go no further with involution than this. If, however, he will learn the symbolism of algebra he may easily express and work with 4th, 5th, 6th, even with nth powers. He should be shown that while he can solve equations in one, two, and perhaps in three unknowns with graphical pictures, *i.e.*, geometrically, the great power he gains by mastering the algebraic way enables him to go right on easily to the solution of simultaneous equations in 4, 5, 6, and even n unknowns. He should be made to feel that while arithmetic would enable him, by a slow process of feeling about, to find *one* solution of many problems, algebra, if he will learn its language and method, will lead him directly not to one, but to *all possible* solutions. It will thus enable him to know when he has solved his problem *completely*. These and similar gains of power over quantitative problems are the real reasons why the educated man of today cannot afford not to know algebra. Let teachers perform this professional duty well and the foes of algebra as a school subject will be confined to those who are ignorant of it. The one who has learned the subject will then regard it as the *emancipator of quantitative thinking*.

It is desired to call particular attention to the introductory pages on Reasons for Studying Algebra, and to Suggestions on Problem-solving on page 113, and to the careful treatment of factoring. The treatment of the function notion, on pages 50-56, will appeal to many teachers. It will be noted also that this elementary course is divided into half-year units.

The problem and exercise lists are full, varied, and carefully chosen. Teachers who employ supplementary lists of exercises with the regular text should not require pupils to try to solve all the problems and exercises given here. These lists are made full and varied to afford choice and range of material. Great care has been exercised to cover all the standard difficulties of first-year algebra, for this book makes its primal task to teach good algebra.

This text is to be followed presently by a second course on Intermediate Algebra. The two together will cover the standard requirements of secondary algebra.

The pleasant task now remains to acknowledge the assistance the authors have received from Mr. John DeQ. Briggs of St. Paul Academy, St. Paul, Minn.; from the Misses Ellen Golden and Estelle Fenno of Central High School, Washington, D. C.; and from Professor H. C. Cobb of Lewis Institute, Chicago, all of whom read and criticized the proofs of the book. Their criticisms and suggestions have resulted in numerous improvements.

May this book find friends amongst teachers and pupils, and a deserving place amongst the influences now making for the improvement of the educational results of high school algebra.

THE AUTHORS.

Chicago, September, 1916.

CONTENTS

FIRST HALF-YEAR

CONTENTS

INTRODUCTION

REASONS FOR STUDYING ALGEBRA

The high school pupil should become convinced, as early as possible, that there are strong reasons why he should learn algebra. The kind of work the pupil will do and his consequent sense of its actual value to him, depend so largely on the approval he gives to its study that it seems worth while, even before beginning it, to consider the reasons for studying algebra.

ALL TASKS REGARDED AS PROBLEMS TO BE SOLVED

Whether a pupil continues in school or leaves early for the work of life, he will soon learn that the best way to deal with the questions and difficulties that arise, is to regard them as *problems* to be solved, and to attack them as such. How to learn his lessons, to write a composition, to do an experiment, to debate a question, to win in a contest, to do anything the first few times, are familiar *problems* to the high school pupil.

How to earn more and waste less, to manage affairs more economically, to get more out of and to put more into life, how to conduct household affairs more economically, to learn to appreciate and to understand more of the really good and true in books and in life, are actual problems to every right-minded man and woman. Right living is little more than solving a continuous chain of problems. The question for every young person should be, "How far can I advance in the problem-book of the great world before the problems get too hard for me?"

IMPORTANT TO ACQUIRE POWER AND SKILL IN PROBLEM WORK

Clearly then, it is of great importance to learn what it means to solve a problem and to acquire whatever skill we may in the art of problem-solving, and this too, not merely because our teacher, or our parents want us to do so, but for our own sakes purely. In an especial sense algebra teaches the *tactics* and the *technique* of problem-solving. The tools by which both the science and the art are wrought out are the algebraic number and the algebraic equation. To be without the ability to use the equation skillfully is to be without the ability to do much problem-thinking. Power to use the equation with skill and insight is the main part of the equipment of an accurate thinker, and algebra is essentially the science and the art of the equation.

TWO REASONS WHY ALGEBRA SHOULD BE STUDIED BY ALL

Every person who has his way to make in the world must succeed or fail in his struggle with life's problems. The world's problems are harder than those of algebra, but the best way to acquire ability to grapple with harder problems is first to get some skill with easier ones. Algebra starts with comparatively simple difficulties that gradually increase in complexity as one's skill grows, to difficulties great enough to tax the powers of even the brightest pupils.

For two reasons at least, the problem-solving of algebra is easier than that of everyday life.

In the first place, the language of algebra makes reasoning easier than does any other language men have yet devised.

Before algebraic language was invented, the ancient mathematicians used ordinary words and sentences in the problems they attempted. The form of their work, which was largely sentence-making, is now called *rhetorical* algebra. It never amounted to much as a problem-solving instrument.

Mathematicians later made use of abbreviated words, phrases, and even sentences that occurred frequently in problems, of initial letters, suggestive symbols, and thus formed what is now called abbreviational, or *syncopated* algebra. This was a real advance, and a very fair sort of algebra now developed as the need for it came along, and men grew interested in it. But it was still cumbersome, and men continued trying to improve it in this way and that, until finally after many centuries, they hit upon the modern form of writing algebraic numbers and relations. From this time forth, *symbolic* algebra, as we now know it, step-by-step, but rapidly, grew up. The advance in mathematics and mathematical science that soon followed is almost incredible. Thus the history of mathematics shows two things, viz.:

1. *That advance in mathematical thought depends greatly on the kind of language employed, and*

2. *That the language of modern symbolic algebra is the most powerful aid to precise thinking that the world has yet found.*

Every civilized race uses this language today. Of all existing languages of the world it is best entitled to be called the universal language of man.

In the second place, algebraic problems have definite answers, so that the beginner may always have a complete *check* on his thinking during the apprenticeship-period while he is necessarily somewhat doubtful about its reliability. On the other hand, the problems of life have no answers, or the answers are of the general nature of success or failure in one's enterprises. With the latter there is no chance to go back and correct errors before the errors have resulted fatally. This is a strong reason why algebra is a good early training in problem-study and problem-strategy. We can do hard things by virtue of the power and skill acquired in doing similar, but easier, things.

ALGEBRA NOT CREATED FOR A MERE SCHOOL DISCIPLINE

It thus appears that algebra was not created, as pupils are sometimes prone to think, merely as a severe discipline for school boys and girls. Algebra was formed through the united efforts of a long succession of scientific men to devise a tool and technique for solving the problems of science that arose from age to age — problems that no known subject or device could conquer. It was created as a necessity to win even the little scientific knowledge the race acquired from age to age. After algebra had revealed the desired solutions, sometimes another mathematical subject was found capable of yielding a solution also, but algebra was usually the pioneer, and it is only rarely that any science furnishes easier and more reliable ways of solving problems than algebra. To be ignorant of algebra is to be deprived of the most effective problem-solving engine yet invented. Why not seize the opportunity to acquire some mastery over this powerful tool? The beginnings of the subject are easily within the comprehension of the twelve-year old boy or girl.

ALGEBRA IS FUNDAMENTAL TO ALL MATHEMATICAL SCIENCES

One of the strongest reasons for studying algebra is that it is fundamentally necessary to so many fields of higher scientific work. Aside from a little elementary geometry, almost no mathematics beyond the simplest arithmetic is possible without a knowledge of algebra. To attempt to get on in mathematics without algebra is verily "to try to walk without feet." Perhaps the most widely useful mathematical subject within reach of high school students is trigonometry. Trigonometry is the science of the triangle, and is made up very largely of compact practical rules, or laws, expressed in the language of algebraic formulas and equations. The transformations of these formulas that lead to the most

recently: "O, that I knew enough algebra to enable me to understand the formulas of Kent's *Engineers' Pocket-Book*, to be able to make proper substitutions in these formulas, and to know the meaning of the results!" It is the weakness of their problem-solving ability that men of practical affairs seem most to regret. These men often contend that much of what they had to study in the high school has been of little or no use to them, but that they could not have been given too much mathematics for the work they have since had to do. They tell us the leaders today are not the great orators and charming talkers of a generation ago, but the *mathematized* thinkers. It is the latter, they tell us, that are carrying off the prizes of this commercial and industrial age.

Let boys and girls who have not yet lost the opportunity to profit by school-work in mathematics make this study as profitable as possible to themselves, by taking up the fundamental subject of algebra with energy and determination. Dismiss the idea, if you hold it, that you are studying this subject as a favor to your teacher or parents. Embrace and cherish the true idea that you are studying it for your own benefit, to raise your own efficiency, and that you are only cheating yourself if you do poor work. The chances are many to one that the tasks of after-life will be found to make stronger demands on your problem-solving ability than algebra requires. Do not forget that algebra is in a peculiar sense the subject which can best develop and perfect ability of this type. Therefore take up the work vigorously the first day, never relaxing your efforts to master the subject until the last lesson is learned.

ELEMENTARY ALGEBRA

First Half-Year

CHAPTER I

NOTATION IN ALGEBRA. THE EQUATION

NOTATION

1. The power of algebra is due mainly to its language and its symbols. You have already made some start with this language, for everything you have correctly learned about the language and the symbols of arithmetic holds good also in algebra. But because algebra is a sort of *general* arithmetic, it adds something to the language and symbols of arithmetic and employs them more *generally* than arithmetic does. Perhaps the most important things for the beginner to keep in mind from the outset are that what the algebraic language talks about and what the algebraic symbols stand for are *numbers* and **number relations.** Though the book or the teacher may talk about algebraic expressions, or *quantities*, or monomials, or polynomials, it is important to remember that all these terms, and many others, are only other names for **numbers.**

Algebra, like arithmetic, treats of numbers. It adds, subtracts, multiplies, and divides numbers, raises them to powers, and extracts their roots.

2. Notation is the method of expressing numbers by figures or letters.

In arithmetic, numbers are represented by ten Arabic characters called *digits* or *figures*. Thus,

$$453 = 400 + 50 + 3$$

The Number. The whole of the number is the *sum* of the parts represented by the several digits.

Representing Numbers. In algebra, numbers are represented by figures, by letters, and by a combination of both.

3. Products. When letters and figures are written together in algebra, their *product* is indicated.

Thus, $4a$ means 4 times a, and $7ab$ means $7 \times a \times b$

If a number is the product of two or more numbers, those numbers are **factors*** of the product.

The numbers represented by the digit and the letters in $7ab$ are therefore factors of the whole number, $7ab$.

4. Using Algebraic Language. The expression, $5x$ yards, means 5 times the number of yards represented by x.

Exercise 1

1. What is meant by the expression, $9x$ quarts? $12y$ cents? $3n$ miles? $8x$ square feet?

2. If n represents a certain number, what does $4n$ represent? $9n$? bn?

3. If x represents the price of a yard of silk, what does $5x$ represent? $8x$? $12x$?

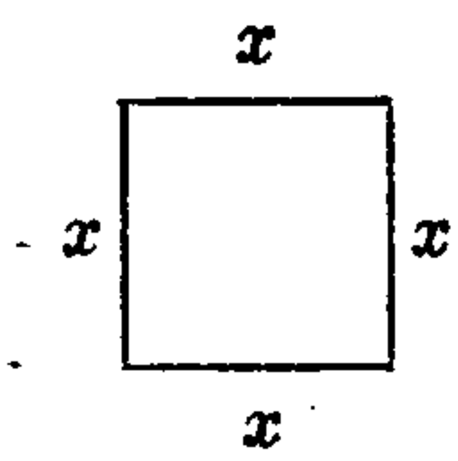
4. A boy bought 8 oranges at b cents apiece. How many cents did he pay for them?

5. What is meant by the expression, $6x$ yards? $4a$ dollars? $8y$ bushels? $7x$ square rods?

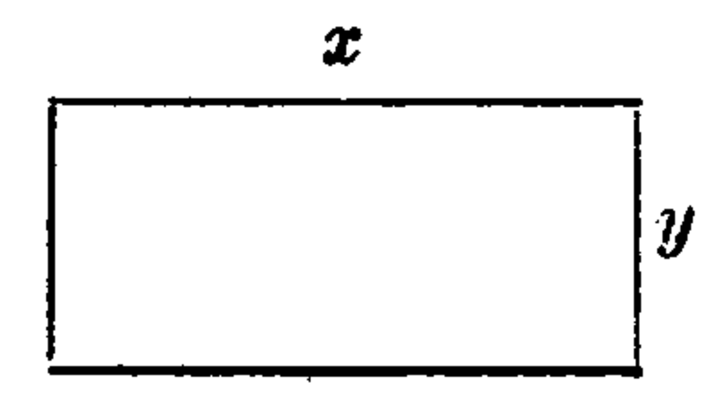
*The word *factor* means *maker.* The factors of a number are its *makers by multiplication.*

6. If a man works for n dollars a day, how much does he earn in 8 days? In x days?

7. If a square is x inches on each side, what does $4x$ represent? What does xx represent?



8. How many square inches are there in a rectangle x inches long and y inches wide?



9. If you are c years old today, how old is your father who is three times as old?

10. If n represents a certain number, what represents 6 times the number? m times the number?

5. Algebraic Signs. The signs of addition, subtraction, multiplication, and division mean the same as in arithmetic.

6. Indicating Multiplication. Multiplication is often indicated in algebra by placing a dot between the factors. Thus,

$$5\times a\times c=5ac=5\cdot a\cdot c$$

7. Indicating Division. Division is often indicated by writing the dividend over the divisor in the form of a fraction.

Exercise 2

1. Indicate the sum of 8 and 7. Of x and 5. Of a and b. Of x, y, and z. Of $2a$, $3b$, and 12.

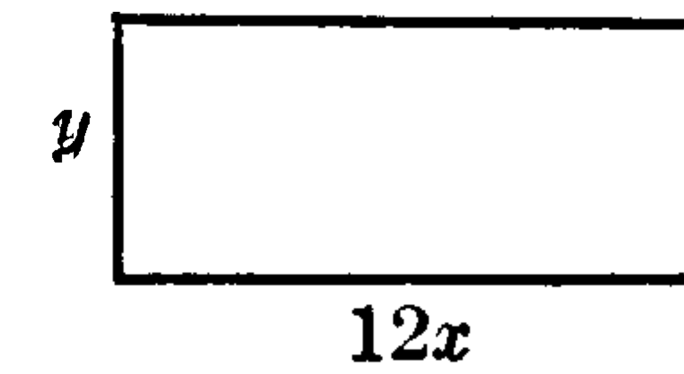
2. A man is n years old today. How old was he 7 years ago? Eighteen years ago?

3. If a man has p sheep in one field and q sheep in another, how many has he in both fields?

4. What is meant by the expression, $7x$ feet? $9y$ square feet? $n+8$ days?

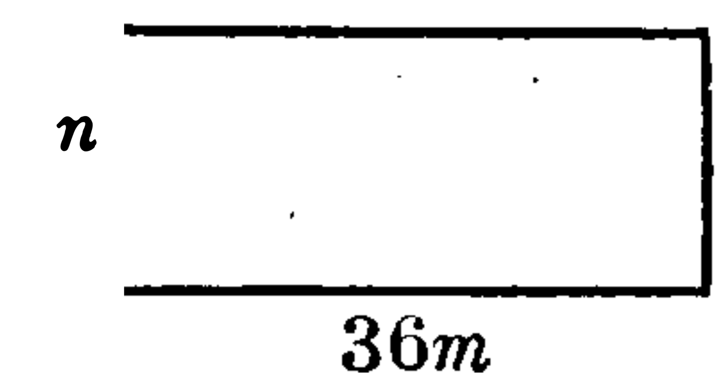
5. When n represents an odd number, what will represent the next larger odd number?

6. What represents the number of square inches in a rectangle x feet by y inches?



7. If n represents an even number, what will represent the next smaller even number?

8. How many square inches are there in a rectangle m yards long and n inches wide?



9. If the sum of two numbers is x and the larger number is y, what is the smaller number?

10. If a rectangular piece of land is x rods long and y rods wide, what does $2x+2y$ represent?

11. Indicate in two ways the product of 5 and x. Of b and y. Of 5, a, and b. Of n, x, and 3.

12. A man bought x cows at $35 apiece and had $85 left. How much money did he have at first?

13. Indicate the difference between a and b when a is greater than b. When b is greater than a.

14. A boy had a cents. He earned b cents and then spent 8 cents for candy. How many cents did he have left?

15. A boy has m quarters and n dimes. What expression represents the number of cents he has?

16. What represents the number of square yards in a ceiling x feet long and y feet wide?

17. What will denote the number of acres in a rectangular field L rods long and W rods wide?

18. A man sold a horse for b dollars and gained c dollars. How much did the horse cost him?

19. A man bought x sheep at m dollars a head and y lambs at n dollars a head. What did all cost him?

20. A boy bought n apples at x cents apiece and sold them at y cents apiece. If he gained, what was his gain?

Since 5 times any number + 4 times the same number = 9 times that number, $5a+4a=9a$, also

$$9x+4x=13x \qquad 4a+a=5a \qquad 4n+2n+n=7n$$

21. If a man gets x dollars for corn and $4x$ dollars for wheat, how much does he get for both?

Since 9 times any number − 3 times the same number = 6 times that number, $9b-3b=6b$, also

$$10a-3a=7a \qquad 5x-x=4x \qquad 8b-3b-b=4b$$

22. If the larger of two numbers is $8s$ and the smaller is $3s$, what represents the difference between the two numbers?

23. A has n sheep. B has twice as many as A, and C has twice as many as A and B. How many have all?

24. If a set of harness costs x dollars, a carriage $3x$ dollars, and a horse $5x$ dollars, what do all cost?

25. If a man has $5y$ dollars and spends x dollars for a suit of clothes, how many dollars has he left?

THE EQUATION

8. The **sign of equality** is =. It indicates that the numbers between which it is placed are equal.

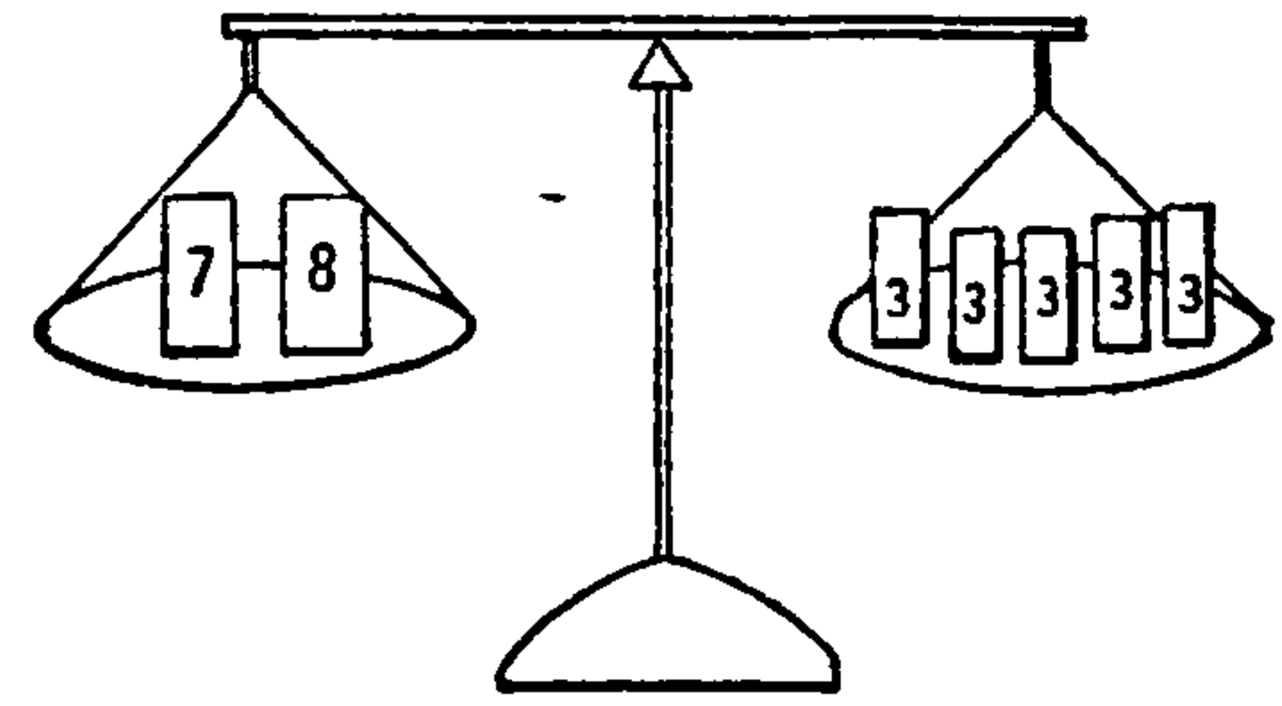
The expression $9x-5=2x+9$, means that the difference between $9x$ and 5 is the same as the sum of $2x$ and 9.

9. An **equation** is an *expression of equality* between two equal numbers. Thus,

$$8+7=5\times 3 \qquad 8x+6=3x+36 \qquad 4n+2n=54$$

10. The **first member** of an equation is the number on the left of the sign. The **second member** is the number on the right.

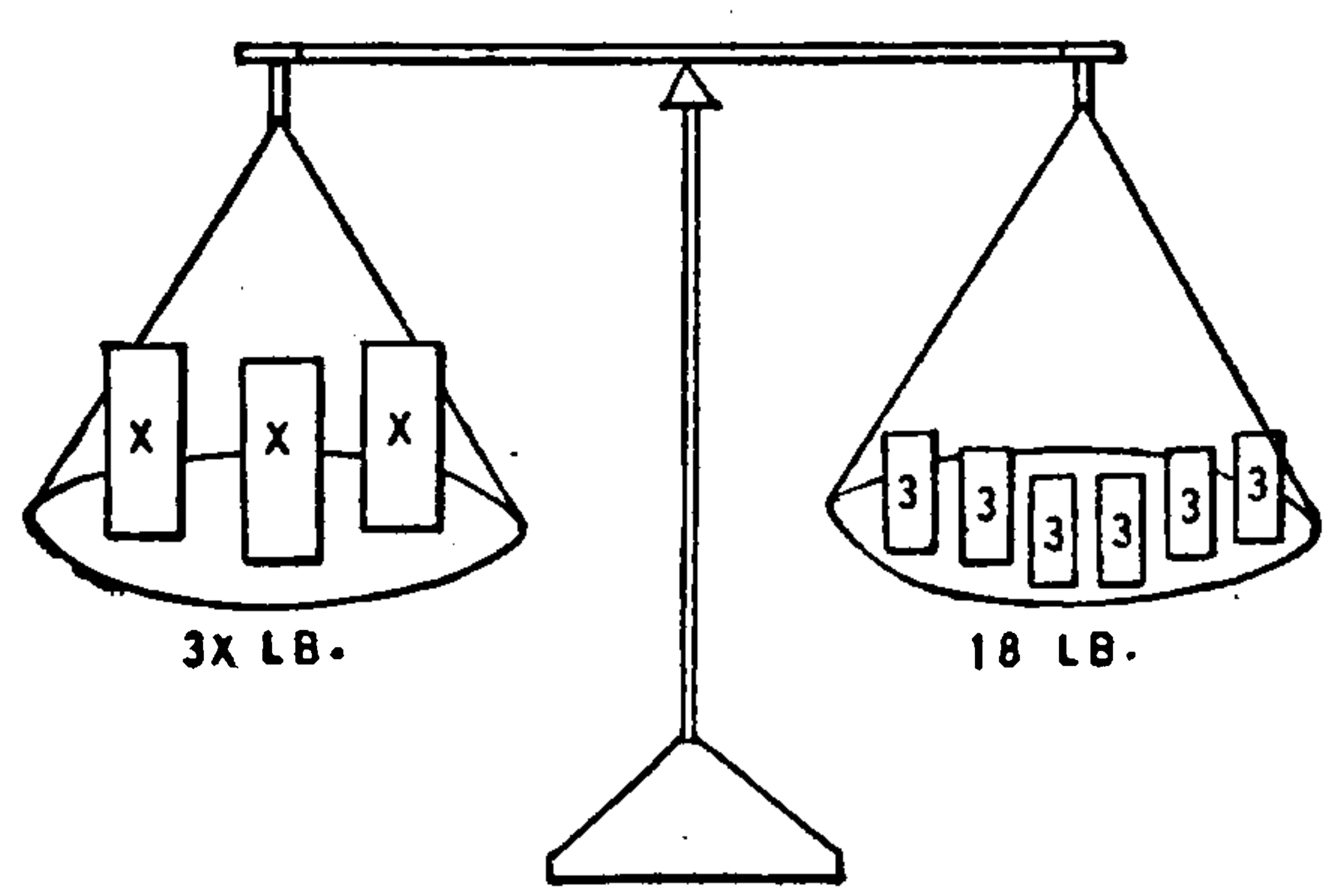
11. The equation expresses *balance of values* just as the horizontal position of the bar of the balances shows *balance of weights.* To put = between two number expressions is to say that if the numbers were weights and the expressions in the two members were represented by proper weights, one in each pan, the balance-bar would stand horizontal.



$7+8=5\times 3$

12. The value of any letter in a number expression is the number or numbers which it represents.

If 3 unknown weights of x lb. each in one pan are balanced by 6 weights of 3 lb. each in the other pan, we may say $3x=18$. Leaving $\frac{1}{3}$ of the weights on each side on the pans, and removing the rest, the bar will remain horizontal, or we may say, $x=6$ lb. That is, the bar can be horizontal when the pans are loaded one with $3x$ lb. and the other with 18 lb. only if $x=6$ lb.

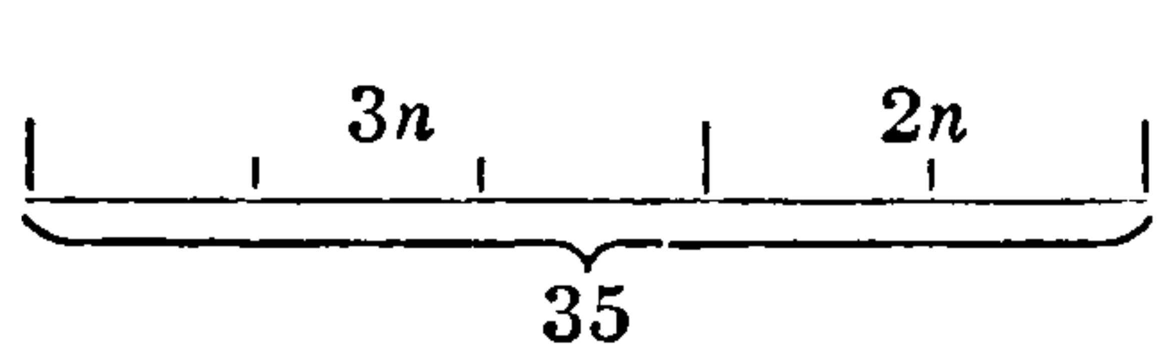


$3x=18$

But without troubling with the balance, by merely applying the division principle that equal numbers divided by the same number give equal numbers, to the equation $3x=18$, we find $x=6$. In this way algebra makes *reasoning* take the place of the *weighing apparatus.*

In the equation, $3x=18$, since $3x$ means 3 times x, $3x$ and 18 are equal only when x represents 6.

In the equation, $3n+2n=35$, since $3n+2n$ is $5n$, $3n+2n$ and 35 are equal only when n represents 7; or think thus:



, whence $n=7$.

In algebra when the reason for a change in an equation is asked, the pupil is expected to quote or to cite an axiom that justifies the change.

16. Give the reason for the conclusion in each of the following:

1. $x=7$ and $y=4$; then $x+y=11$
2. $c=21$ and $d=6$; then $c-d=15$
3. $a=x$ and $b=3$; then $a-b=x-3$
4. $c=2n$; then $10c=20n$
5. $d=32$; then $\frac{d}{8}=4$
6. $x=7$ and $y=7$; then $x=y$
7. $3y=27$; then $y=9$
8. If $n=5$; then $8n=40$
9. If $m=9$ and $n=4$; then $mn=36$
10. If $m=28$ and $n=4$; then $\frac{m}{n}=7$
11. If $x-3=5$; then $x=8$
12. If $\frac{a}{16}=5$; then $a=80$
13. If $mn=7n$; then $m=7$

Exercise 3

1. Solve $8x-3x+2x-x=30$.

$$8x-3x+2x-x=30$$
$$6x=30$$

By the division axiom, $x=5$

Checking, $40-15+10-5=30$

or $30=30$

Always check or test the value of the unknown number after it is found, by substituting it for the unknown number in the given equation.

Solve and check:

2. $6x-2x+3x=49$

3. $5n-2n+4n-n=48$

4. $5s+6s-3s=48$

5. $9a-3a-2a+a=45$

6. $8y-4y-2y=24$

7. $8b+7b-b-4b=55$

8. $7x+2x-3x=54$

9. $4n-3n+6n-n=72$

17. Solving a problem is the process of finding the values of the unknown numbers involved in the problem.

In arithmetic the unknown numbers are found by one or more of the fundamental processes.

In algebra the unknown numbers are represented by letters and their values are found by the use of equations.

Solving a problem in algebra involves three steps: *notation, statement, solving an equation.*

Exercise 4 — Solving Problems

1. The sum of two numbers is 252, and the larger number is 6 times the smaller. Find the numbers.

Notation, $\begin{cases} \text{Let } s = \text{the smaller number;} \\ \text{then } 6s = \text{the larger number.} \end{cases}$

Hence $s+6s$ and 252 are two number expressions, each of which represents the *sum* of the two numbers.

Statement, $s+6s=252$

The **notation** is the representation in algebraic symbols of the unknown numbers in the problem.

The **statement** is the expression of the conditions of the problem in one or more equations.

Solving the equation, $\begin{cases} 7s=252 \\ s=36 \\ 6s=216 \end{cases}$

To check, substitute in the *statement.* Thus,

$$36+216=252, \text{ or } 252=252$$

Even the statement itself may be wrong. To test whether this is the case, substitute in the conditions of the problem itself.

2. The sum of two numbers is 846, and the larger number is 8 times the smaller. Find the numbers.

3. Seven times a certain number plus 6 times the number minus 8 times the number equals 175. Find the number.

18. Obtaining Statements of Problems. To obtain the statement in a problem is to translate the conditions of the problem into an equation.

DIRECTIONS FOR MAKING STATEMENTS AND SOLVING PROBLEMS

1. *Let any appropriate letter represent one of the unknown numbers to be found.*

2. *From the conditions of the problem express, in terms of the same letter, the other unknown numbers.*

3. *Find two number expressions that represent the same number and place them equal, forming an equation.*

4. *Solve the equation and determine whether the result satisfies the conditions of the problem.*

Exercise 5

1. One number is 5 times another, and the difference between them is 48. Find the numbers.

Notation, $\begin{cases} \text{Let } \quad s = \text{the smaller number;} \\ \text{then } 5s = \text{the larger number.} \end{cases}$

Hence, $5s-s$ and 48 are two number expressions, each of which represents the *difference* between the numbers.

Statement, $5s-s=48$

Solving this equation, $s=12$ and $5s=60$

Checking, $60-12=48$, or $48=48$

2. A is six times as old as B, and the difference between their ages is 75 years. Find B's age.

3. In a school of 855 pupils there are twice as many girls as boys. How many girls are there?

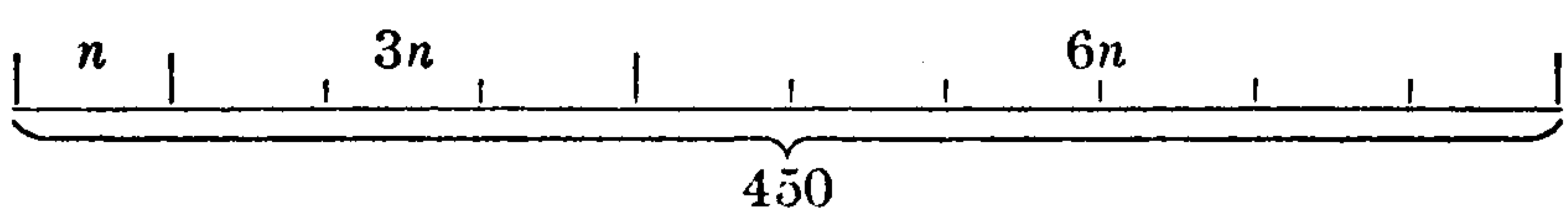
4. A earned five times as much as B. If B earned \$648 less than A, how much did both together earn?

5. The length of a rectangle is 3 times its width, and the perimeter is 224 inches. Find the dimensions.

19. Letters Represent Numbers. In solving problems, always let the letter represent some *number*. It must not represent money, but a *number* of dollars or cents; not time, but a *number* of days or hours; not weight but a *number* of pounds or ounces; not distance, but a *number* of miles, rods, or other units of measure.

Exercise 6 — Problems

1. A horse, carriage, and harness cost \$450. The carriage cost 3 times as much as the harness, the horse twice as much as the carriage. Find the cost of each.

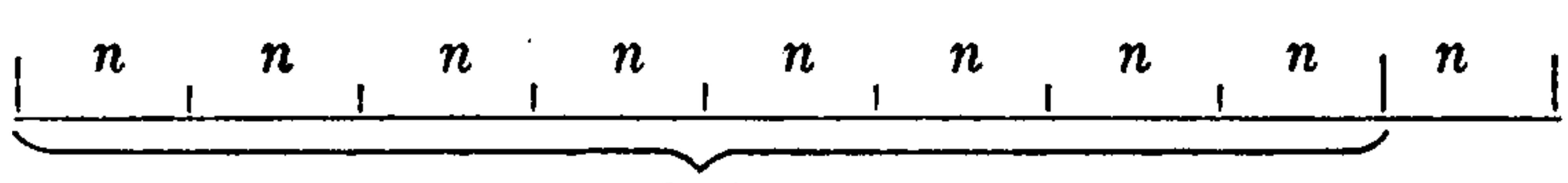


Let n = the number of dollars the harness cost;
then $3n$ = the number of dollars the carriage cost;
and $6n$ = the number of dollars the horse cost.

Hence $n+3n+6n$ and 450 are two number expressions, each of which represents the cost of all.

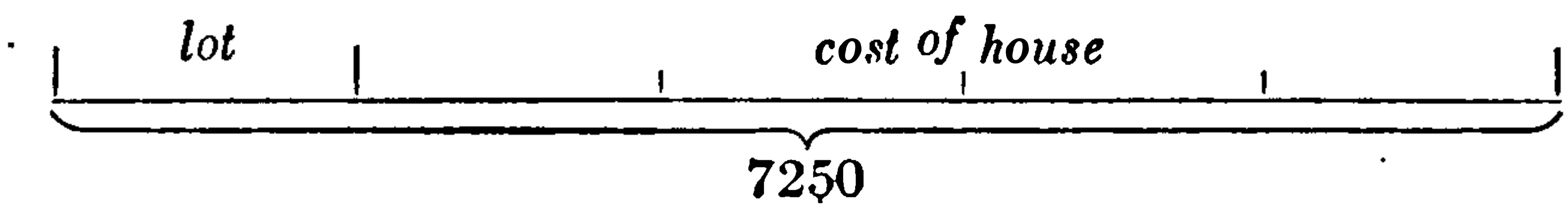
$$n+3n+6n=450$$

2. One number is 9 times another, and the difference between them is 624. Find the numbers.

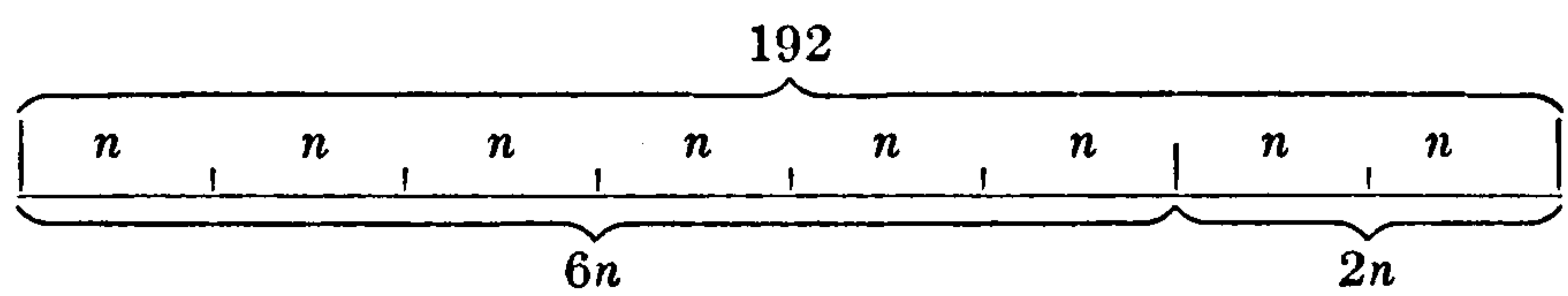


3. A has twice as many sheep as C, and B has 4 times as many as C. If all have 665, how many has B?

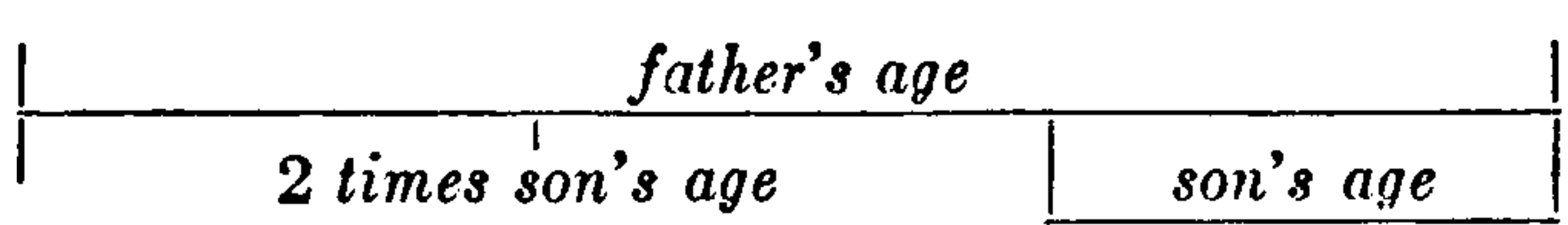
4. A house and lot cost $7250, the house costing 4 times as much as the lot. Find the cost of each.



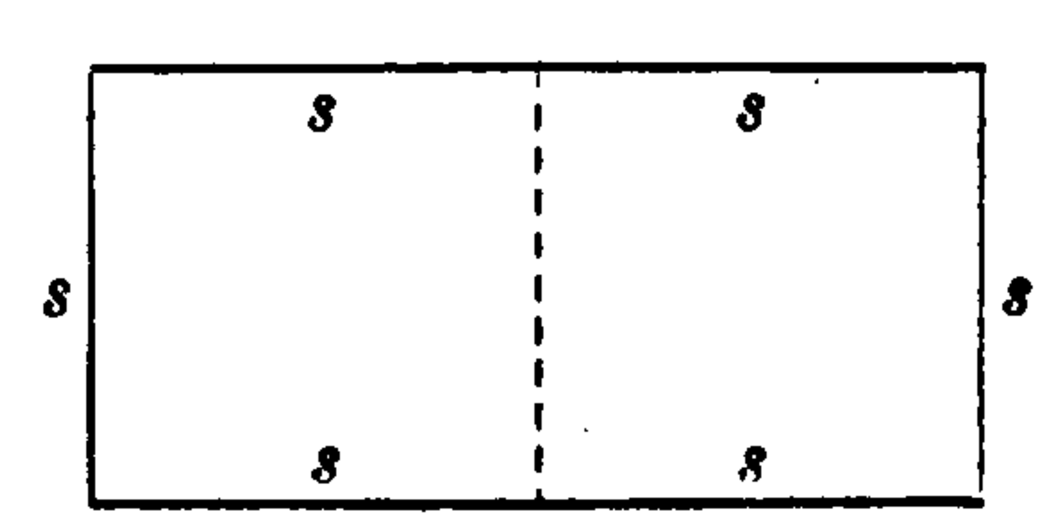
5. If twice a number is added to six times the same number, the sum is 192. Find the number.



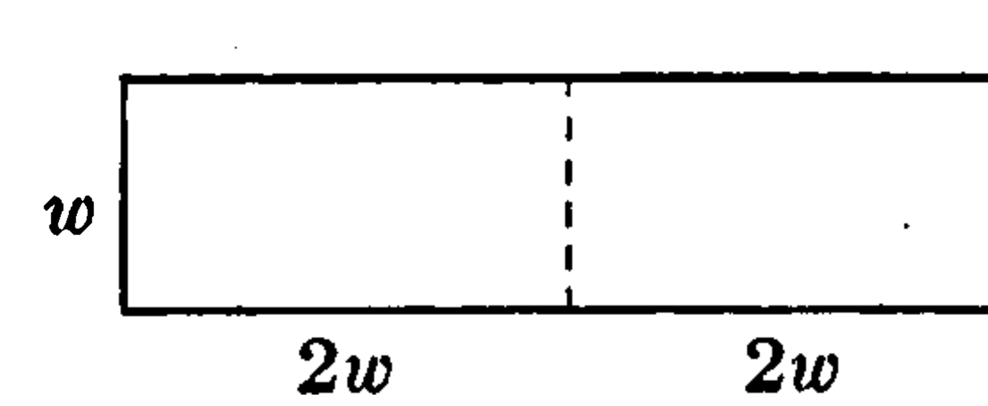
6. The sum of the ages of father and son is 96 years, and the difference between their ages is twice the son's age. What is the father's age?



7. A rectangle formed by placing two equal squares side by side has a perimeter of 270 feet. Find the side of each square and the area of the rectangle.



8. If two rectangles of the same width and twice as long as wide are placed end to end, the perimeter of the rectangle formed is 180 inches. Find their dimensions.



9. One number is 4 times another, and 4 times their difference is 576. Find the numbers.

10. A man sold a horse and carriage for $340, receiving 3 times as much for the horse as for the carriage. How much did he get for the carriage?

11. The sum of two numbers is 322, and their difference is 5 times the smaller. Find the larger number.

12. A, B, and C own 840 sheep. A owns 3 times as many as B, and C owns twice as many as A and B together. How many do A and C together own?

13. A's age exceeds B's by 3 times B's age, and the sum of their ages is 75 years. Find A's age.

14. In a mixture of 228 bushels of corn and oats there are twice as many bushels of corn as of oats. How many bushels of oats are there in the mixture?

15. A number increased by 3 times itself, 4 times itself, and 5 times itself is 650. Find the number.

16. A man sold some lambs at $3 a head and three times as many sheep at $5 a head, receiving $324 for all of them. How many of each did he sell?

17. The length of a rectangle is 4 times its width, and the perimeter is 280 yards. Find the dimensions.

18. A, B, and C own 600 acres of land. B owns 3 times as many acres as A, and C owns half as many acres as A and B together. How many acres have B and C?

19. A merchant paid $50 for two pieces of silk of equal value, paying 80¢ a yard for one piece and $1.20 a yard for the other. How many yards were in each piece?

20. Two equal rectangles whose length is 3 times the width, if placed end to end, form a rectangle whose perimeter is 196 inches. Find the length of each rectangle.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS. DEFINITIONS

POSITIVE AND NEGATIVE NUMBERS

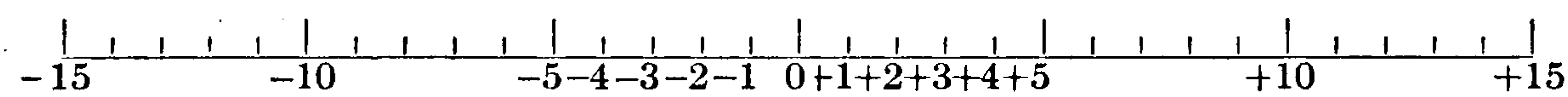
20. Numbers of Arithmetic. The only relation of numbers considered in arithmetic is the relation of size.

A boy starts from 0, takes 12 steps toward the *right*, then turns, and takes 7 steps toward the *left*. How far is he then from the *starting-place*, 0?

In arithmetic we would solve this problem thus:

$$12-7=5$$

But suppose after taking 12 steps to the right and turning back, he had taken 20 steps toward the left. Where would he then be with regard to the starting-point?



An Algebraic Scale

We know that in arithmetic we cannot subtract 20 from 12. Still by using the algebraic scale above, we can easily solve the problem, and learn that the boy will be 8 steps to the *left* of the starting-point, 0. If we agree that the sign $-$, instead of meaning "subtract" shall mean "*go leftward*," we may write:

$$12-20=-8.$$

It will be more complete also to agree that the sign, $+$, instead of always meaning "add," as it did in arithmetic, may mean also "*go rightward*"; hence we write:

$$+12-20=-8,$$

which means "12 steps rightward, followed by 20 steps leftward, leaves one 8 steps *left* of the starting-point."

are used in algebra, and the quality of a number is denoted by the sign $+$ or $-$.

The sign $+$ before a number denotes that it is *positive*, and the sign $-$ that it is *negative*, as $+5$, -6.

24. The **absolute value** of a number is the number of units in it, independent of their quality.

The absolute value of $+9$ is 9.
The absolute value of -8 is 8.

Exercise 7

Let us consider distance *north* from a certain point as *positive* and distance *south* as *negative*.

1. If a man walks north 12 miles one day and north 13 miles the next day, what is the result?

2. If a man walks south 11 miles one day and south 10 miles the next day, what is the result?

3. If a man walks north 14 miles one day and south 10 miles the next day, what is the result?

4. If a man walks north 10 miles one day and south 15 miles the next day, what is the result?

5. If a man walks south 14 miles one day and north 11 miles the next day, what is the result?

6. If a man walks south 10 miles one day and north 17 miles the next day, what is the result?

You have probably answered these six questions as follows: He is 25 miles north of the starting-point; 21 miles south; 4 miles north; 5 miles south; 3 miles south; 7 miles north.

Here are the algebraic solutions of the six problems. Tell how each result is obtained and what it represents.

$+12$	-11	$+14$	$+10$	-14	-10
$+13$	-10	-10	-15	$+11$	$+17$
$+25$	-21	$+\ 4$	$-\ 5$	$-\ 3$	$+\ 7$

The results of uniting these positive and negative numbers show the following principles:

25. *The sum of two numbers with like signs is the sum of their absolute values with the common sign prefixed.*

26. *The sum of two numbers with unlike signs is the difference between their absolute values with the sign of the number having the greater absolute value prefixed.*

Exercise 8

Applying these principles, write the sums in the following examples, giving each its proper sign:

$+23$	$+33$	-31	-41	$+29$	$+19$
$+15$	-14	-16	$+17$	-14	-37
$+75$	$+83$	-67	$+43$	-28	-73
$+68$	-38	-49	-82	$+74$	$+37$
$+85$	$+39$	-49	$+93$	-65	-34
$+78$	-75	-68	-45	$+29$	$+73$

27. Double Meaning of + and −. Thus it appears that the signs + and − are used in algebra to denote *quality* of numbers as well as to denote *operations*.

Exercise 9—Problems with Positive and Negative Numbers

Assign quality to the numbers in these problems, solve them algebraically, and interpret the results:

1. A man's property amounts to $18,750 and his debts to $23,250. Find his net debt or property.

2. A merchant gains $2365 one year and loses $1790 the next year. Find the net gain or loss.

3. If a man travels east 58 miles one day and west 73 miles the next day, what is the net result?

4. A man's annual income is \$3675 and his expenses \$2395. How much does he save annually?

5. If a ship sails north 53 miles one day and south 39 miles the next day, what is the net result?

6. A real estate dealer gains \$1465 on one sale and \$2375 on another. Find the result of both sales.

7. Draw a line representing a thermometer scale; mark the zero point, 24°, and $-12°$. What is the difference between the $+24°$ and the $-12°$?

8. If the weight of a stone is regarded as positive, what would represent the weight of a balloon?

9. If a speculator makes \$2765 one month and loses \$2875 the next month, what is his net gain or loss?

10. If a stone weighs 34 pounds, and a balloon pulls upward with a force of 8 pounds, what is the combined weight of both, if they are fastened together?

DEFINITIONS

28. A **system of notation** is a system of symbols by means of which numbers, the relations between them, and the operations to be performed upon them can be more concisely expressed than by the use of words.

29. **Algebraic notation** is the method of expressing numbers by figures and letters.

30. An **algebraic expression** is the representation of any number in algebraic notation.

31. A **term** is a number expression whose parts are not separated by the sign + or −, thus,

$$2a \times 4b, \quad 3ab, \quad xy, \quad 5ax, \quad \text{and} \quad \frac{5x}{3y}$$

32. A **monomial** is an expression of *one* term. A **polynomial** is an expression of *two* or *more* terms, as,

$$2a+4b-3c-5d$$

The signs + and − between the terms of a polynomial may be regarded as signs of *operation* or of *quality.*

When monomials and the first term of a polynomial are written without any sign before them, they are *positive.*

33. A **binomial** is a polynomial of *two* terms. A **trinomial** is a polynomial of *three* terms.

34. A **coefficient** of a term is any factor of the term which shows how many times the other factor is taken as an addend. Thus,

$$4n=n+n+n+n \qquad 4ax=ax+ax+ax+ax$$

Coefficients are distinguished as *numerical* or *literal,* according as they are expressed in figures or letters.

In the two terms above, 4 is the numerical coefficient.

Any other factor of $4ax$ may be regarded as the coefficient of the product of the remaining factors.

Observe that $4a+a=a+a+a+a+a=5a$

This shows that when no numerical coefficient is expressed, the numerical coefficient is considered to be 1.

35. **Similar terms** are terms which do not differ, or which differ only in their numerical factors, as,

$5xy$, xy, and $8xy$; $3ab$ and $5ab$; or $4ax$, ax, and $7ax$

36. **Dissimilar terms** are terms that are not similar, as

$4ab$, ax, $3bc$; $3ac$, $4xy$; $2xy$, xz, $3yz$

37. **Partly Similar Terms.** Terms that have a common factor are said to be *partly similar,* or *similar with respect to that factor.*

Thus, ax, $4x$, and bx are similar with respect to x; and $5xy$, axy, bxy, and $4cxy$ are similar with respect to xy.

38. The **value of an algebraic expression** is the number it represents when some particular value is assigned to each letter in the expression.

Substitute 1 for a, 2 for b, 3 for c, 4 for d, in the following expression and simplify the result:

$$2ab + 3bc + 5cd - 4bd =$$
$$2\cdot 1\cdot 2+3\cdot 2\cdot 3+5\cdot 3\cdot 4-4\cdot 2\cdot 4=$$
$$4 + 18 + 60 - 32 = 50$$

Exercise 10

Find the value of each of the following expressions when $a=1$, $b=2$, $c=3$, $d=4$, $e=0$, $m=\frac{1}{2}$, $n=\frac{1}{3}$:

1. $bcd-9n-4a+6am$
2. $6am+9a+2bc-3n$
3. $6b-adm+5bc-6n$
4. $4ad-6n+2dm-2b$
5. $bcd-5e-4m+3cn$
6. $8b-4am+9bn-2a$
7. $8a+6mn-2b+bcd$
8. $8ad-7e-bm+6dn$
9. $5cd-8m+9a-6cn$
10. $7a+9bd-8m+9an$
11. $4bc+7d-9n+7ab$
12. $5bd+ac-4bm+6n$
13. $cdm+3cn+ab-7e$
14. $6bn+5e-6m+8ad$
15. $9n+cdm+de-2ab$
16. $cd-acn+8m+3em$

CHAPTER III

ADDITION

ADDITION OF MONOMIALS

39. Addition is the process of uniting two or more numbers into one number.

40. The **addends** are the numbers to be added; the **sum** is the number obtained by addition.

41. To Add Similar Terms. In adding $5 \cdot 6$ and $3 \cdot 6$ in arithmetic, the two products, which are 30 and 18, are found and then added.

Since 5 times 6 plus 3 times 6 is 8 times 6, they may be added also by adding the coefficients of 6, thus

$$5 \cdot 6 + 3 \cdot 6 = 8 \cdot 6$$

42. Adding Indicated Products. Algebraic terms, which are indicated products, can be united into one term only by the latter method. For example:

1. A school hall is l yards long. I go through it 6 times on Monday and 14 times on Tuesday. How many yards do I travel through the hall on both days?

Monday,	$6l$ yards
Tuesday,	$14l$ yards
Both days,	$20l$ yards

2. The tickets for an entertainment were t cents each. George sold 34 and Mary 28 tickets. Find the total receipts from the sales of George and Mary.

George,	$34t$ cents
Mary,	$28t$ cents
Both,	$62t$ cents

ADDING SIMILAR TERMS

43. *The sum of two similar terms is the sum of their coefficients with the common letters affixed.*

Whether the terms have like or unlike signs, the sum of the coefficients is found by §§ 25 and 26.

Exercise 11

Give at sight the sum of each of the following:

1. $4\cdot 3$	**2.** $4a$	**3.** $8x$	**4.** $-7b$	**5.** $-3c$
$5\cdot 3$	$-5a$	$-3x$	b	$-5c$
6. $8\cdot 5$	**7.** $9a$	**8.** $-9x$	**9.** $9b$	**10.** $-6c$
$6\cdot 5$	$-3a$	$4x$	$-b$	$-4c$
11. $6\cdot 7$	**12.** $2x$	**13.** $-4b$	**14.** n	**15.** $-3c$
$7\cdot 7$	$-7x$	$5b$	$-4n$	$-7c$
16. $5a$	**17.** $-7x$	**18.** $2b$	**19.** $-n$	**20.** $-8c$
$6a$	$6x$	$-6b$	$6n$	$-6c$
21. a	**22.** $9x$	**23.** $-7b$	**24.** $-8n$	**25.** $-4c$
$7a$	$-3x$	$8b$	n	$-9c$
26. $5a$	**27.** $-9x$	**28.** $9b$	**29.** $-n$	**30.** $-7c$
$9a$	$8x$	$-2b$	$4n$	$-5c$
31. $7a$	**32.** $4y$	**33.** $-6b$	**34.** $-7n$	**35.** $-6c$
$8a$	$-9y$	$7b$	n	$-9c$

44. Rule.— *Find the* **algebraic sum** *of the coefficients, and to that result affix the common letters.*

6.	a	**7.**	$6x$	**8.**	$-9b$	**9.**	$2n$	**10.**	$-8c$
	$8a$		$-9x$		$3b$		$-4n$		$-c$
	$2a$		x		$7b$		$8n$		$-9c$

11.	a	**12.**	$-7x$	**13.**	$5b$	**14.**	$6n$	**15.**	$-3c$
	$5a$		$4x$		$-6b$		$-9n$		$9c$
	a		$-6x$		$9b$		$2n$		$-6c$

16.	$5a$	**17.**	$3x$	**18.**	$-b$	**19.**	$6n$	**20.**	$-9c$
	a		$-8x$		$-b$		$-n$		$4c$
	$7a$		$2x$		$8b$		$8n$		$-8c$

ADDING DISSIMILAR TERMS

48. Dissimilar terms cannot be united into one term. The addition can only be indicated by writing them in succession in any order, each preceded by its own sign, as here shown:

$$\begin{array}{r} 3ac \\ -bc \\ 2bd \\ \hline 3ac-bc+2bd \end{array} \qquad \begin{array}{r} 5a \\ -2b \\ -3a \\ \hline 2a-2b \end{array} \qquad \begin{array}{r} -3bc \\ -4ac \\ -2b \\ \hline -4ac-3bc-2b \end{array}$$

We write a positive term first, if there is one. If all the terms are negative, any one of them may be written first.

Exercise 13

Give at sight the sum of each of the following:

1.	$3a$	**2.**	b	**3.**	$-2x$	**4.**	$-2n$	**5.**	$-5c$
	$2x$		$-2c$		y		$-3x$		$-c$

6.	$4a$	**7.**	$2a$	**8.**	$-n$	**9.**	$2a$	**10.**	$5x$
	$3x$		$-b$		$-3x$		$-b$		$-6y$
	a		$-c$		$4n$		$-5c$		$-4x$

11.	**12.**	**13.**	**14.**	**15.**
$2a$	$8x$	$-x$	$-7n$	$2a$
b	$-4x$	$-3y$	$-n$	$-c$
$5x$	$-5x$	$4x$	$4n$	$-5c$
16.	**17.**	**18.**	**19.**	**20.**
$4a$	$4x$	$-3b$	$3a$	$-7c$
$2b$	$-2y$	$6b$	$-c$	$-9c$
a	$-x$	$-4b$	$-7n$	$-8c$
21.	**22.**	**23.**	**24.**	**25.**
$7a$	$-5x$	$4b$	$-3n$	$-6c$
a	$3y$	n	$7n$	$4c$
$4a$	x	$-6x$	$-5n$	$-5c$
26.	**27.**	**28.**	**29.**	**30.**
$3x$	$-2x$	$7b$	$-2n$	$-5a$
a	$5x$	$-4c$	$8n$	$-2b$
$5a$	$-3x$	b	$-6n$	$-4y$
31.	**32.**	**33.**	**34.**	**35.**
a	$-5x$	$-3a$	$3n$	$-a$
$3n$	$3x$	b	$-9n$	$-5c$
x	$-7x$	$-a$	$7n$	$-c$
36.	**37.**	**38.**	**39.**	**40.**
$4a$	x	$-6b$	$-4n$	c
a	$-y$	$-b$	$2a$	$9c$
$2a$	$4z$	$-7b$	n	$-c$
41.	**42.**	**43.**	**44.**	**45.**
y	$-5x$	$6a$	n	$-a$
$7x$	$3x$	$-b$	$-3n$	$-3c$
x	$-7x$	$2n$	n	$-4a$
46.	**47.**	**48.**	**49.**	**50.**
$4a$	$3b$	$-7y$	$5c$	$-7x$
a	$-5b$	y	b	$3x$
$6a$	$-b$	$6y$	$-2c$	$-x$
a	$9b$	$-4y$	$2b$	$6x$

Simplify the following:

51. $4a+2a+a+5a$

52. $3n+8n+n+2n+6n$

53. $2x+x-7x+8x$

54. $5b-2b-6b+b+9b$

55. $5c-6c+c+4c$

56. $6y-y+9y-7y+5y$

57. $7a-3a-a-2x$

58. $7n+5n-n-4n+3a$

59. $8x-4x-3x-y$

60. $4b+9b+7b-8b-b$

61. $6y+8x-9y-5x$

62. $6a-7b-4a+3b+a$

ADDITION OF POLYNOMIALS

49. Addition of polynomials proceeds much as addition of monomials, as the two following illustrations show:

1. The stairway of a school has 3 flights, of a, b, and c steps, respectively. A boy goes up and down the stairway 3 times on Monday, 5 times on Tuesday, 4 times on Wednesday, 6 times on Thursday, and 4 times on Friday. How many steps does he take on the stairs during the week?

Monday,	$6a+6b+6c$	steps
Tuesday,	$10a+10b+10c$	steps
Wednesday,	$8a+8b+8c$	steps
Thursday,	$12a+12b+12c$	steps
Friday,	$8a+8b+8c$	steps
Sum,	$44a+44b+44c$	steps

2. At a money-changer's are offered for exchange:

At one time, 52 marks, 35 francs, 12 pounds;
At another, 18 marks, 26 francs, 24 pounds;
At another, 22 marks, 15 francs, 18 pounds.

The exchange value of a mark being m cents, of a franc f cents, and of a pound l cents, find the total exchange value of the foreign currency in cents.

First time,	$52m+35f+12l$	cents
Second time,	$18m+26f+24l$	cents
Third time,	$22m+15f+18l$	cents
Sum,	$92m+76f+54l$	cents

50. To add polynomials, *write similar terms in a column and add each column, beginning at the left.*

Thus,

$$\begin{array}{rrrrrrr}
5ab & +3ac & -2bc & +3bd & +5xy & -7xz & \\
2ab & -\ ac & & -5bd & +2xy & & \\
 & 5ac & -\ bc & & & +7xz & \\
ab & & +3bc & & -4xy & & +6 \\
\hline
8ab & +7ac & & -2bd & +3xy & & +6
\end{array}$$

51. A check on algebraic work is another operation which tends to prove the first result correct.

52. Checking Addition by Substitution. Addition may be checked by substituting any number in place of the letters and determining whether the *sum of the values* of the addends equals the *value of the sum.*

The following shows how addition of polynomials may be checked by substituting 1 for each letter.

$$\begin{array}{rcl}
\textbf{Work} & & \textbf{Check} \\
5a-9b+7c & = & 5-9+7=3 \\
a+8b-6c & = & 1+8-6=3 \\
3a-4b+3c & = & 3-4+3=2 \\
\hline
9a-5b+4c & = & 9-5+4=8
\end{array}$$

The sum of the values of the addends is 8, and the value of the sum of the polynomials is also 8.

Observe that when 1 is substituted for each letter, the value of each term is the numerical coefficient.

In checking or verifying algebraic processes, any number may be substituted for each letter. To avoid large numbers, it is well to substitute small numbers; but substituting 1 *checks only the coefficients* and should not, *in general,* be done.

Exercise 14

1. Add $4a-3n+2x$, $5n-4x+5$, $-7a-4n+7x$, $2a+6n+6$, $n-x-14$, and $5a-4n-3x+4$.

2. Add $5b+3c-6d$, $c-2b+3d$, $-4d-5c+3b$, $6d-4c-7b$, and $2c+d+4b$, and check.

3. Add $4a-3b+5c$, $2c-2b+d$, $-4d-8a+7b$, $3c+4a+3d$, and $2b-d-7c$, and check.

4. Add $7x-5y+3z$, $3y-8-5z$, $-4y+6z-5x$, $6y-2x-8z$, and $4z+3y+8$, and check.

5. Add $3ax+4by-2xy$, $5by-7xz+6xy$, $2ax-3xy-9by$, and $7xz+xy-by$, and check.

6. Add $2x-4y+3z$, $-5z+y-6x$, $3y+z+5x$, $-8y+4x-4z$, and $7y+3z-4x$.

7. Add $7ac-an+3nx$, $5ax+4an-6nx$, $2nx-3an-5ac$, and $an-5ax-nx$, and check.

8. Add $5a+6b-7c$, $4c-3b+5$, $-2c+5b-8a$, $4c-7b-9$, and $-b+2c+6a+5$.

9. Add $5ax+3bx-2cx$, $3dx-4ax+5cx$, $4cx-7bx-3dx$, and $6bx-8cx+ax$.

10. Add $8ab-6bc+4ac$, $5ad-5ab-7ac$, $4bc-ad+5ac$, and $3bc-3ad-3ab$.

11. Add $4an-7bn+5ab$, $4bn-7ac-6an$, $3an+6ac-9ab$, and $4bn-ac+4ab$, and check.

12. Add $6x-7y+5z$, $4y-u-3z$, $-2u+6y-5x$, $4z-5y+4u-x$, and $-6z+2x-3u$.

13. Add $4ab-2ac+4bc$, $-5ac-2ab+6bc$, and $ab-2ac$.

14. Add $4a-7b-5c$, $3c-7a-d$, $-5b+3d-2c$, $8b-2d-5+c$, and $4b+2c+4a+5$.

15. Add $5xy-4xz+3yz$, $2xz-2xy-7yz$, $3xz-xy+9yz$, and $-8yz+6xy-xz$, and check.

CHAPTER IV

SUBTRACTION. SYMBOLS OF AGGREGATION

53. Subtraction is the process of finding one of two numbers when their sum and the other number are known.

54. The **minuend** is the number that represents the sum; the **subtrahend** is one of the addends of the minuend.

55. The **difference,** or **remainder,** is the number which added to the subtrahend gives the minuend.

SUBTRACTION OF MONOMIALS

1. A thermometer reads $+13°$, and four hours previously it read $-7°$. Through how many degrees and in what direction had the top of the mercury changed meanwhile?

Present reading,	$+13°$
Previous reading,	$-\ 7°$
The change,	$+20°$, obtained by subtracting $-7°$ from $13°$.

2. Starting from a stair-landing a boy goes up 17 steps, and drops his pencil, which rolls down to the landing, across the landing, and on down to the 6th step below the landing, where it stops. The steps are a inches high. How far and in what direction must the boy go to get to the step where the pencil lies?

Calling upward $+$ and downward $-$, the boy

arriving at	$-\ 6a$
starting from	$+17a$
goes	$-23a$, meaning $23a$ inches downward.

In these cases we have been subtracting signed numbers. Let us now learn the general plan of subtracting such numbers.

SUBTRACTING SIMILAR TERMS

56. The following examples represent all cases **in addition** with reference to signs and relative values of addends:

$5a$	$3a$	$-5a$	$-3a$	$-5a$	$+5a$	$-3a$	$3a$
$3a$	$5a$	$-3a$	$-5a$	$3a$	$-3a$	$5a$	$-5a$
$8a$	$8a$	$-8a$	$-8a$	$-2a$	$2a$	$2a$	$-2a$

Write examples *in subtraction,* using the above sums as minuends and one addend as subtrahend, as follows:

$8a$	$8a$	$-8a$	$-8a$	$-2a$	$2a$	$2a$	$-2a$
$3a$	$5a$	$-3a$	$-5a$	$3a$	$-3a$	$5a$	$-5a$
$5a$	$3a$	$-5a$	$-3a$	$-5a$	$5a$	$-3a$	$3a$

By the definition of subtraction, the difference or remainder in each case must be the other addend.

Show that the correct result might have been obtained in each case *by changing the sign of the subtrahend and adding.*

57. Principle.— *Subtracting any number is equivalent to adding a number of equal absolute value but opposite quality.*

58. Rule.—*Conceive the sign of the subtrahend changed from + to − or from − to + and proceed as in addition.*

The change of sign should always be made mentally.

Exercise 15 — Subtracting Similar Terms

Give remainders in the following orally:

1. $9a$ $4a$	**2.** $-4x$ $6x$	**3.** $-3b$ $-8b$	**4.** $7n$ $2n$	**5.** $-11c$ $-3c$
6. $5a$ $4a$	**7.** $3x$ $-7x$	**8.** $-4b$ $-b$	**9.** n $6n$	**10.** $-10c$ $4c$

Exercise 16 — Subtracting Monomials

Give remainders in the following orally:

1. $\begin{array}{r} 3a \\ b \\ \hline \end{array}$	**2.** $\begin{array}{r} -4x \\ -7x \\ \hline \end{array}$	**3.** $\begin{array}{r} -4a \\ -2n \\ \hline \end{array}$	**4.** $\begin{array}{r} -5n \\ 7n \\ \hline \end{array}$	**5.** $\begin{array}{r} 4x \\ -\ y \\ \hline \end{array}$
6. $\begin{array}{r} 7a \\ 8a \\ \hline \end{array}$	**7.** $\begin{array}{r} -7a \\ -2c \\ \hline \end{array}$	**8.** $\begin{array}{r} -6b \\ b \\ \hline \end{array}$	**9.** $\begin{array}{r} 2n \\ -3x \\ \hline \end{array}$	**10.** $\begin{array}{r} 7c \\ -6c \\ \hline \end{array}$
11. $\begin{array}{r} a \\ 2b \\ \hline \end{array}$	**12.** $\begin{array}{r} -3x \\ -9x \\ \hline \end{array}$	**13.** $\begin{array}{r} -\ a \\ -3x \\ \hline \end{array}$	**14.** $\begin{array}{r} -3n \\ 6n \\ \hline \end{array}$	**15.** $\begin{array}{r} 5x \\ -3y \\ \hline \end{array}$
16. $\begin{array}{r} x \\ 9x \\ \hline \end{array}$	**17.** $\begin{array}{r} 3a \\ -2b \\ \hline \end{array}$	**18.** $\begin{array}{r} -2b \\ 7b \\ \hline \end{array}$	**19.** $\begin{array}{r} -3n \\ -4n \\ \hline \end{array}$	**20.** $\begin{array}{r} 5c \\ -\ c \\ \hline \end{array}$
21. $\begin{array}{r} a \\ 5b \\ \hline \end{array}$	**22.** $\begin{array}{r} -\ x \\ -9x \\ \hline \end{array}$	**23.** $\begin{array}{r} -\ a \\ -4b \\ \hline \end{array}$	**24.** $\begin{array}{r} 4n \\ -3n \\ \hline \end{array}$	**25.** $\begin{array}{r} -\ c \\ 8c \\ \hline \end{array}$
26. $\begin{array}{r} 3x \\ 6x \\ \hline \end{array}$	**27.** $\begin{array}{r} 5a \\ -2c \\ \hline \end{array}$	**28.** $\begin{array}{r} -3b \\ -7b \\ \hline \end{array}$	**29.** $\begin{array}{r} -5n \\ -6a \\ \hline \end{array}$	**30.** $\begin{array}{r} -2x \\ -5y \\ \hline \end{array}$
31. $\begin{array}{r} 4a \\ 9a \\ \hline \end{array}$	**32.** $\begin{array}{r} x \\ -6y \\ \hline \end{array}$	**33.** $\begin{array}{r} -4b \\ 7b \\ \hline \end{array}$	**34.** $\begin{array}{r} -2n \\ -4a \\ \hline \end{array}$	**35.** $\begin{array}{r} -5c \\ -\ c \\ \hline \end{array}$
36. $\begin{array}{r} a \\ 7x \\ \hline \end{array}$	**37.** $\begin{array}{r} -7x \\ 5x \\ \hline \end{array}$	**38.** $\begin{array}{r} -2a \\ -2b \\ \hline \end{array}$	**39.** $\begin{array}{r} -4n \\ -9n \\ \hline \end{array}$	**40.** $\begin{array}{r} -8c \\ c \\ \hline \end{array}$
41. $\begin{array}{r} 3a \\ 5a \end{array}$	**42.** $\begin{array}{r} -\ x \\ 8y \end{array}$	**43.** $\begin{array}{r} 6b \\ -8b \end{array}$	**44.** $\begin{array}{r} -4n \\ 5c \end{array}$	**45.** $\begin{array}{r} 7c \\ -9c \end{array}$

SUBTRACTION OF POLYNOMIALS

60. **1.** Subtract 7 dollars, 3 quarters, 8 dimes from 16 dollars, 7 quarters, 12 dimes.

Letting c be the number of cents in a dollar, q the number of cents in a quarter, and d the number of cents in a dime, we write:

$$\begin{array}{ll} \text{From} & 16c+7q+12d \\ \text{Take} & \underline{\;\;7c+3q+\;\;8d} \\ \text{Difference,} & \;\;9c+4q+\;\;4d \end{array}$$

2. From $5ab-4ac+3bc$ bushels of grain, $4ab-6ac+2cd$ bushels were sold. How many bushels remained?

$$\begin{array}{lllll} \text{Minuend,} & 5ab & -4ac & +3bc & \\ \text{Subtrahend,} & \underline{4ab} & \underline{-6ac} & \underline{} & \underline{+2cd} \\ \text{Difference,} & ab & +2ac & +3bc & -2cd \end{array}$$

61. Rule.—*Write the polynomials, similar terms in a column. Beginning at the left, subtract as with monomials.*

Subtraction is checked by determining whether the difference between the values of minuend and subtrahend is equal to the value of the remainder. Observe the work below:

$$\begin{array}{lcl} \textbf{Work} & & \textbf{Check} \\ 5ab-4ac+3bc & = & 10-12+18 \qquad =16 \\ \underline{4ab-6ac \qquad +2cd} & = & \underline{\;\;8-18 \qquad +24=14} \\ ab+2ac+3bc-2cd & = & \;\;2+\;6+18-24=\;\;2 \end{array}$$

The above example in subtraction has been checked by substituting 1 for a, 2 for b, 3 for c, and 4 for d.

It is now plain that subtracting is finding *what number must be added to the subtrahend to give the minuend.* Hence, another good check on subtraction is to *add the subtrahend and difference and see if the sum is the minuend.*

Exercise 17 — Subtracting Polynomials

Solve the following and check the first nine:

1. From $8ab-5c+4d-8$ subtract $4d+3ab-12-6c$.

2. Subtract $5ay-z+9ax+14$ from $4ax+6ay-z+8$.

3. From $6bc-5b+8de+f$ subtract $8de+10+5bc-5b$.

4. Subtract $4ac+3bd-2bc-8be$ from $4ac+2bd-10be$.

5. From $4cx+7by-xy-9$ subtract $7by-10+3cx+xy$.

6. Subtract $4ax-4xy+7ab$ from $4ax-2ac-3xy+8ab$.

7. From $ax-7ay+3xy-2z$ subtract $4xy-7ay-7+ax$.

8. Subtract $3ab+6-7ac-ax-am$ from $4ab-6ac-am$.

9. From $6am-4an+4ar-7rs$ subtract $12+6am-12an$.

10. From the sum of $3a+2b-3c+d$ and $2d+2a-4b$ subtract $3b-5+3d+4a+3c$.

11. From $4x-3y+2z-u$ subtract the sum of $3z+2x-6-4y$ and $-2z-x+y-2u+6$.

12. Subtract the sum of $2y+2b-5x-3a$ and $3x-6b+3y+4a$ from $2a-3b-2x+4y$.

13. From the sum of $3c-2d-5e+2f$ and $8e-4d-3f-6c$ subtract $5e-5c-f-3d$.

14. Subtract $2b-2c+d-2a$ from the sum of $2a-5b+2c-2d$ and $4b+3d-3a-3c$.

15. From the sum of $3x+2y-z+2u+8$ and $3z-4x-10-5y-4u$ subtract $2z-3-5x-3u-2y$.

16. From $5ab+2ac-3bc$ subtract the sum of $2bc+3ac+bd$, $4ab-2bd-4bc$, and $bd-4ac-ab$.

17. From the sum of $4x+y-2z$ and $4u+3y-7x-2z$ subtract $4u+4y-5z+5-3x$.

18. What number must be added to $-4a+6b-8c$ to give 0? To give $8a+4b-4c$?

19. From $4ab-3ac+2bc$ subtract the sum of $3bc+bd-ac$, $3ab-3bd-bc$, and $bd-2ac-ab$.

20. From the sum of $3a-2x+5$ and $4x+2y-4$ subtract the sum of $3x-2a+3$ and $y-2x-2$.

21. If $x=5a-3b+4c$, $y=3a-2b-3c$, $z=a+b+6c$, find the value of $x-y-z$.

22. What number must be subtracted from $2ab-3ac-5bc$ to give $ac-5bc+2ab-3bd$? To give 0?

23. Subtract $2a-3b+4$ from 7, $2b-3a+3$ from unity,* $a-2b+2$ from zero, and add the three results.

SYMBOLS OF AGGREGATION

62. The product 8×14 can be shown thus: $8(10+4)$, which means $8\times 10+8\times 4=80+32=112$.

This use of the symbol ( ), called a **parenthesis,** is of aid in learning rapid mental calculation, thus:

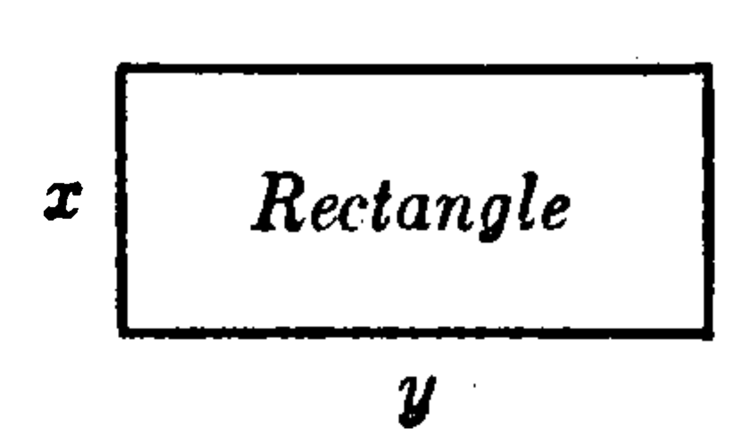
$$7\times 25=7(20+5)=140+35=175$$
$$6\times 49=6(40+9)=240+54=294$$
$$9\times 68=9(60+8)=540+72=612, \text{ etc.}$$

63. A man walks north 5 miles an hour for 2 hours and then south along the same road 3 miles an hour for 2 hours. How far is he then from the starting-point?

The answer to this problem may be written thus:

$$2(5-3)=2\times 2=4$$

Show that the perimeter of a rectangle x wide and y long may be written: $2(x+y)$ or $2x+2y$, or $x+y+x+y$ and that $2(x+y)=2x+2y$.



*Unity means 1.

64. In a series of the four operations, the multiplications and divisions are to be performed first. Thus,

$$\begin{aligned}&8+7\times3-6+16\div2+5-5\times3-8\div2=\\&8+\ 21\ -6+\ \ 8\ \ +5-\ 15\ -\ \ 4\ \ =17\end{aligned}$$

In such a series the *terms* are the parts separated by the signs + and −. The above example contains seven terms. When such expressions are to be simplified or reduced, each term should be first simplified or reduced.

When it is desired to perform the operations of a series in any order other than the one mentioned above, it is necessary to use some *symbol of aggregation*.

65. The **symbols of aggregation** are the *parenthesis* ( ), the *brace* { }, the *bracket* [ ], and the *vinculum* ———.

These mean that the operations indicated *within them* are to be performed before the operations *upon them*; in other words, that the expressions within them are in each case to be regarded as one number. Every part within the symbol is affected by the operation indicated upon the symbol. Observe the following:

$$18-9-4=5 \qquad\qquad 15\times12-8=172$$

$$18-(9-4)=13 \qquad\qquad 15\times\overline{12-8}=60$$

$$\begin{aligned}&216-(24-36\div4)\times4-(4+6\times3-\overline{35-8\times4})=137\\&216-\qquad 60\qquad -\qquad\qquad 19\qquad\qquad =137\end{aligned}$$

Notice the use of the parenthesis in the following:

1. If the smaller of two numbers is $x-7$ and the larger $x-2$, their difference is $(x-2)-(x-7)$.

2. If a rectangle is $x+8$ in. long and $x+3$ in. wide, the area of the rectangle is $(x+8)(x+3)$ square inches.

3. If $\frac{1}{3}$ of the distance between two cities is $x+10$ miles, the whole distance is $3(x+10)$ or $(x+10)3$ miles.

Exercise 18

Remove the symbols of aggregation and then simplify:

1. $465+67\times8-(9\times24+144\div4-\overline{45\div5}\times6)$

2. $764-(245-465\div5)-14\times7+789-540\div9$

3. $238-8\times9-108\div9+754-(84-58)-47\times8$

4. $9\times(48+65)+128\div4-(8\times\overline{12+7}-8\times8)\times7$

66. Operations on Compound Expressions. Symbols of aggregation are much used in algebra to indicate operations on compound expressions.

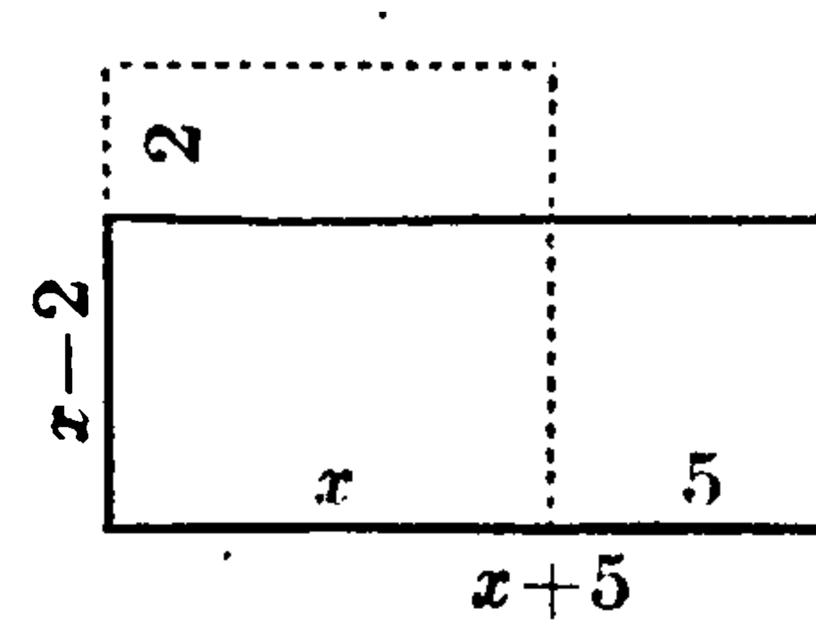
To indicate the subtraction or multiplication of a polynomial, a parenthesis is necessary.

Thus, $x(a+b)$ represents the product of x and $a+b$ and is read x times $a+b$, or $a+b$ times x.

Exercise 19

1. Indicate the subtraction of $x-5$ from $3x+4$. Indicate the product of two binomials.

2. If a man has $8x$ sheep and sells $2x+35$ of them, what will denote the number he has left?



3. What does $(x+5)(x-2)$ represent, if x represents the number of feet on each side of a square?

4. What does $x(x+8)$ represent, if x stands for the number of rods on each side of a square?

5. Represent in two forms 4 times the sum of any two numbers. 5 times the difference of any two numbers.

6. Represent the product of two equal numbers each of which is 8 greater than x.

7. At 85¢ a rod, express in two ways the cost of enclosing a rectangular farm x rods by y rods.

8. If x is any positive integer greater than 5, is $x-5$ greater or less than $x-3$? Show why.

9. What is the equation which tells that the difference between $x-9$ and $x-4$ is a?

10. If the difference between $x-12$ and $x-8$ is n, what is the value of n?

11. If x is any positive integer, when is ax greater than x? When is ax less than x?

12. How many trees are there in an orchard, if there are 20 more trees in a row than there are rows?

13. Write $3a$ times the product of two binomials divided by the product of $a+b$ and $a-b$.

14. Indicate how many acres there are in a rectangular field $x-8$ rd. wide and $x+10$ rd. long.

15. What may represent the product of 4 numbers, if any 2 of them in order differ by the same number?

16. Write an expression of 3 terms, each term containing one or more compound factors.

17. At $40 an acre, what is the value of 3 farms containing x, $x+20$, and $x-5$ acres, respectively?

18. Represent the product of two unequal numbers, part of each number being x.

19. What is the area of 3 equal rectangles, the width of each being x in. and the length 6 in. greater?

20. The area of a square x in. long is the same as that of a rectangle $x+6$ in. by $x-4$ in. Express as an equation.

21. How much does a boy earn, if the number of cents he gets per day exceeds the number of days he works by 20?

Exercise 21

Remove the symbols of aggregation in the following and express the results in as few terms as possible:

1. $4a-b-(a-2b+c)$
2. $3x-(-2x+3y)+2y$
3. $3a-(b-c+2a)+b$
4. $4x+3y+(-3x-4y)$
5. $5a-b-\overline{4a+b-c}$
6. $5x-(-2x-4y)-3y$
7. $2a+b-c+(3a-b)$
8. $2x-3y-(-2x-4y)$
9. $3a-\overline{c-b-4a}-b$
10. $4n-3x+(-3n-4x)$

When $x=2a-3b+4c$, $y=3a+2b-5c$, $z=4a-5b-3c$, find the value of each of the following:

11. $x+y+z$
12. $x+y-z$
13. $-x-y-z$
14. $x-y-z$
15. $x-y+z$
16. $-x+y-z$

69. To remove two or more symbols of aggregation, one within another, begin with the outer one.*

$$3a-(a+2b-a-\overline{b-c}+n)$$
$$=3a-a-2b+a+b-c-n$$

It should be noted that the − sign before the b belongs to the vinculum, not to the b. The sign of the b is +.

Removing the outer symbol changes the sign before $\overline{b-c}$ to +, and these two terms are brought down with the same signs.

*Many teachers prefer to begin with the *innermost* symbol of aggregation. Either way becomes easy after a little practice.

It is just about as easy and it is even quicker, to remove *all* symbols of aggregation at once by beginning at the left and bringing each successive term down with its own or the opposite sign according as there is an *even* or an *odd* number of the antecedent minus signs affecting it. Any one of the three ways becomes easy and reliable with a little practice.

Exercise 22

Remove the symbols of aggregation in the following and simplify the results:

1. $6a-(b+5a+c)+b$
2. $2y-3x-(-4x-3y)$
3. $2a-(3b-\overline{a+b-c})$
4. $6x-(-2y+\overline{3y-5x})$
5. $4a-(2b-\overline{a+c}-b)$
6. $5x-(-\overline{2y-4x}-3y)$
7. $5a-(b+\overline{a-2b-a})$
8. $4n-(-3x+\overline{3n-6x})$
9. $3a-(b-\overline{2a+b}-c)$
10. $7x-(-\overline{4y-3x}+3y)$
11. $4a+(b-\overline{a+2b}-c)$
12. $3x+2y-(-2x-4y)$
13. $2a-(c+\overline{b+2a-b})$
14. $4n-(-2x+\overline{3n+2y})$
15. $3b-(a+\overline{3b+a+c})$
16. $2y-(-2x-\overline{3x-3y})$

70. It follows, that in order to enclose two or more terms of a polynomial in a symbol of aggregation preceded by the sign $-$, we must change the signs of the terms enclosed. Thus,

$$ab-ac+bc-cd=ab-(ac-bc+cd)$$

Exercise 23

Enclose the last three terms of each of these polynomials in a parenthesis preceded by a minus sign:

1. $ac-ax+ab+bx$
2. $2x+2y-xy-xz+yz$
3. $ab+bc-ac+ax$
4. $ax-ay-2x+xy-2y$
5. $ax-bx-bc-by$
6. $3a+2b+ax-ab+bc$
7. $an+ab+ac-bc$
8. $2a-ab-ax+bc-2c$
9. $ac-ax-bc+bx$
10. $bc+2a+ac+2x-ac$

ADDITION OF TERMS PARTLY SIMILAR

71. Terms that are partly similar, *i.e.*, similar as to part of the letters only, may be united into one term with a *polynomial* coefficient. Thus,

$$\begin{array}{c} ay \\ by \\ \hline (a+b)y \end{array} \qquad \begin{array}{c} ax \\ x \\ \hline (a+1)x \end{array} \qquad \begin{array}{r} an \\ -2n \\ \hline (a-2)n \end{array}$$

72. Rule.— *Write the dissimilar parts in a parenthesis as the polynomial coefficient of the similar part.*

The above answers are read: "a plus b, times y"; "a plus 1, times x"; and "a minus 2, times n," a slight pause in the reading occurring where the last curve of the parenthesis stands.

Exercise 24

Read the sums of the following:

1. ax bx	2. by $-y$	3. an $-3n$	4. ax x	5. $-by$ cy
6. y by	7. $-ax$ $4x$	8. $-an$ n	9. ab $2b$	10. $-n$ an
11. $3x$ ax	12. an $-cn$	13. $-by$ y	14. ar cr	15. $5x$ $-ax$
16. $3x$ ax x	17. $-4y$ xy $-y$	18. $4x$ $-cx$ x	19. bx x bx	20. xy $-y$ xy
21. ax x ax	22. ay $-y$ $-2y$	23. $3x$ $-2x$ $-nx$	24. ax x bx	25. $-xy$ y $-xy$

SUBTRACTION OF TERMS PARTLY SIMILAR

73. Terms partly similar, *i.e.*, similar as to part of the literal factors, may be subtracted by indicating the subtraction of the dissimilar parts. Thus,

$$\begin{array}{r} ax \\ \underline{bx} \\ (a-b)x \end{array} \qquad \begin{array}{r} by \\ \underline{-cy} \\ (b+c)y \end{array} \qquad \begin{array}{r} n \\ \underline{an} \\ (1-a)n \end{array}$$

74. Rule.—*Write the indicated subtraction of the dissimilar parts in a parenthesis as a polynomial coefficient.*

Observe that the sign of the dissimilar part in the subtrahend is changed from + to −, or from − to +.

Exercise 25

Subtract and read the results of the following:

1. $\begin{array}{r} ay \\ \underline{cy} \end{array}$	**2.** $\begin{array}{r} -bx \\ \underline{-ax} \end{array}$	**3.** $\begin{array}{r} 4a \\ \underline{-4b} \end{array}$	**4.** $\begin{array}{r} ax \\ \underline{x} \end{array}$	**5.** $\begin{array}{r} -an \\ \underline{2n} \end{array}$
6. $\begin{array}{r} ax \\ \underline{2x} \end{array}$	**7.** $\begin{array}{r} n \\ \underline{-an} \end{array}$	**8.** $\begin{array}{r} -ax \\ \underline{xy} \end{array}$	**9.** $\begin{array}{r} ac \\ \underline{cx} \end{array}$	**10.** $\begin{array}{r} -2c \\ \underline{-ac} \end{array}$
11. $\begin{array}{r} b \\ \underline{nb} \end{array}$	**12.** $\begin{array}{r} -y \\ \underline{-xy} \end{array}$	**13.** $\begin{array}{r} -2x \\ \underline{2y} \end{array}$	**14.** $\begin{array}{r} nx \\ \underline{an} \end{array}$	**15.** $\begin{array}{r} ac \\ \underline{-bc} \end{array}$
16. $\begin{array}{r} xy \\ \underline{ax} \end{array}$	**17.** $\begin{array}{r} -x \\ \underline{ax} \end{array}$	**18.** $\begin{array}{r} nx \\ \underline{-xy} \end{array}$	**19.** $\begin{array}{r} c \\ \underline{ac} \end{array}$	**20.** $\begin{array}{r} -ay \\ \underline{by} \end{array}$
21. $\begin{array}{r} bx \\ \underline{ax} \end{array}$	**22.** $\begin{array}{r} y \\ \underline{-by} \end{array}$	**23.** $\begin{array}{r} -3a \\ \underline{-3x} \end{array}$	**24.** $\begin{array}{r} 3y \\ \underline{by} \end{array}$	**25.** $\begin{array}{r} -an \\ \underline{-n} \end{array}$
26. $\begin{array}{r} y \\ by \end{array}$	**27.** $\begin{array}{r} -an \\ nx \end{array}$	**28.** $\begin{array}{r} by \\ -4y \end{array}$	**29.** $\begin{array}{r} bc \\ cy \end{array}$	**30.** $\begin{array}{r} -by \\ y \end{array}$

CHAPTER V

GRAPHING FUNCTIONS. SOLVING EQUATIONS IN ONE UNKNOWN GRAPHICALLY

GRAPHING FUNCTIONS

75. Algebraic Numbers, or Functions. For the present it is convenient to call a number expressed by the aid of one or more letters an **algebraic number** or a **function** of the numbers denoted by the letters. Thus, $2x+3$, n^2-2n-8, $a+b$, $x-y$, etc., are algebraic numbers or functions.

The n^2, in n^2-2n-8, means $n \times n$ and is read *n-square*, just as 5^2 means 5×5 and is read *5-square.*

With every algebraic number or function, such as $3x+5$ (or n^2-2n-8), **two** numbers must be thought about, *viz.*: the **algebraic number** or **function itself** and the **number** x, or n, that it depends on for its value. The number n^2-2n-8 tells us to form a compound number by squaring some simple number (n), subtracting twice the simple number, and then subtracting 8. The two numbers to be thought about are the value of n^2-2n-8 itself, and the value of n, and so for other compound numbers. The number x, n, t, or y, in terms of which the compound number (the function) is expressed, may be called the **independent number.**

In other words, the value of $3x+5$ depends on what x is, and the value of n^2-2n-8 depends on the value of n. The x and the n are the independent numbers.

For the reasons just stated, a number expressed in terms of x, such as $3x+5$, is called a **function** of x, and is written $f(x)$ and read: *function of x.*

Similarly, n^2-2n-8 or any other number expressed in terms of n, may be denoted by $f(n)$ and read: *function of n.*

A function is a number that depends on some other number for its value.

An **algebraic function** is a number whose dependence on another number is expressed in algebraic symbols, as $3x+5$, n^2-2n-8, $a+b$, $x-y$, etc.

In this book the word "function" means *algebraic* function.

A function that depends on two other numbers, as $a+b$, is denoted by $f(a, b)$ and read: *function of a and b*. Thus, also $x-y$ is denoted by $f(x, y)$ and read: *function of x and y.*

The parenthesis, ( ), in the function symbol does not mean multiplication, but is a part of the symbol.

If the letter within the ( ) is replaced by a positive or negative arithmetical number, as in $f(-2)$, the meaning is that the number, -2, is to be substituted for the letter in the function. Thus,

If $f(x)=3x+5$, then $f(-2)=3\cdot -2+5=-6+5=-1$, and if $f(n)=n^2-2n-8$, then $f(5)=5^2-2\cdot 5-8=25-10-8=7$.

Find $f(3)$ if $f(x)=8x-3$.

Find $f(-4)$ if $f(n)=3n+15$.

76. Two very important problems of algebra are:

I. *Knowing the value of the independent number, to find the value of the function;* and

II. *Knowing the value of the function, to find the value of the independent number.*

77. We already know how to solve Problem I.

For example, to find the value of $3x+5$ for $x=4$, we have only to substitute 4 for x in $3x+5$, thus $3\times 4+5$. Reducing, we find $3x+5=17$ for $x=4$, and so also for any other value of x. To find the value of n^2-2n-8 for some value of n, as 5, we substitute 5 for n, thus $5^2-2\times 5-8=7$, to see that $n^2-2n-8=7$ for $n=5$, and so for other values of n.

Thus we know that *to solve the first of the above problems, we have only to substitute the value of the independent number in the function and to simplify.*

78. The second problem occurs very frequently in algebra, *viz.*: To find the value of the independent number when the *value of the function is known.* This is Problem II above and it is the *converse* of Problem I. For example, it is often necessary to solve such problems as:

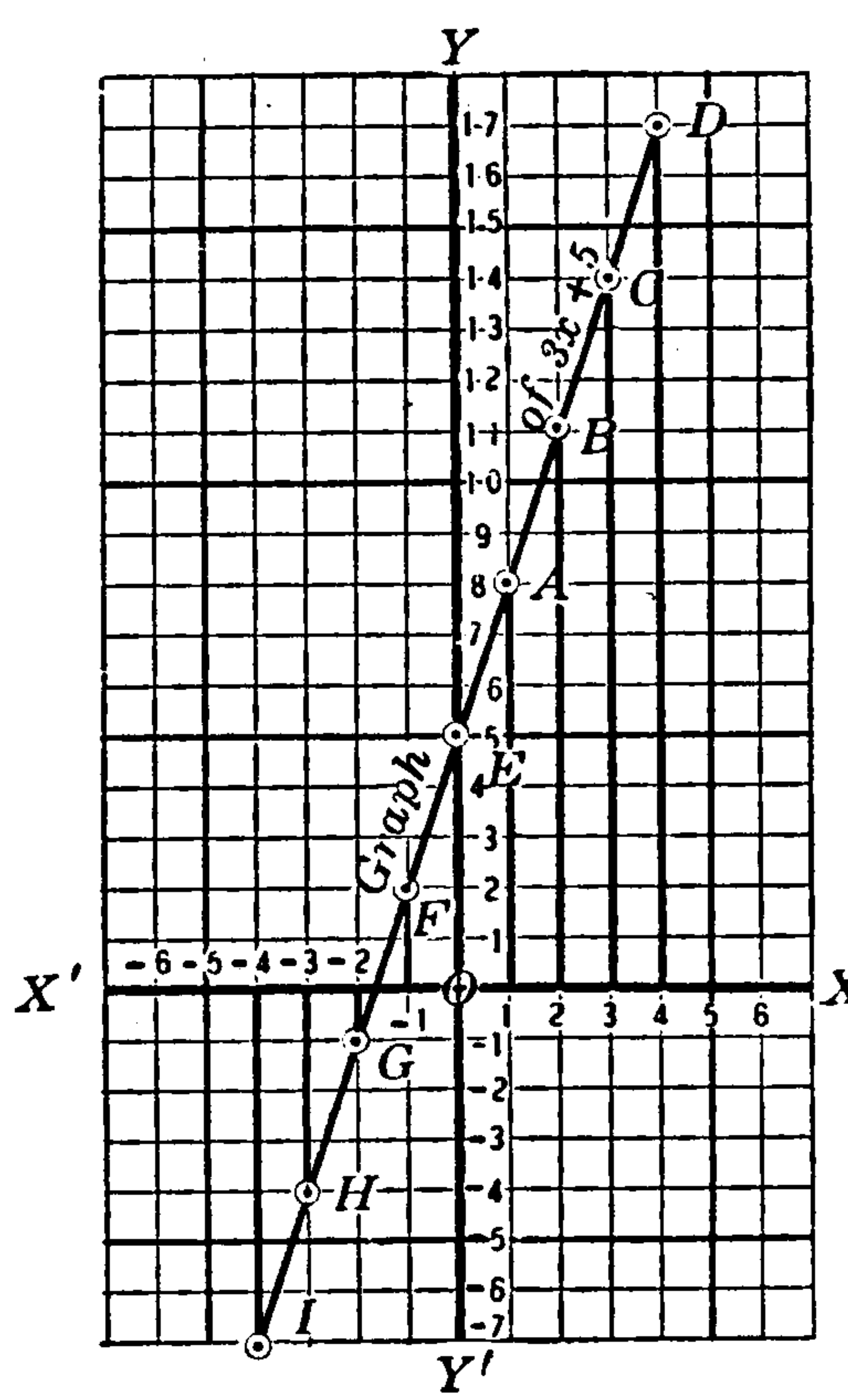
Given $3x+5=8$, to find the value of x, or
Given $n^2-2n-8=7$, to find the value of n.

Such expressions as $3x+5=8$ and $n^2-2n-8=7$ are **equations,** and to solve them means to find what value or values, of x or of n will make $3x+5$ equal to 8, or n^2-2n-8 equal to 7. Consequently, to solve the second problem stated above (§ 76, II), requires a knowledge of the ways of solving equations. We shall first show by means of pictures what it means to solve **equations.**

Let it be kept in mind that **algebraic equations** are made up of **algebraic numbers.**

79. Dependence of an Algebraic Number, or Function. Let us first try to understand the relation that exists between x and $3x+5$.

Draw a vertical and a horizontal algebraic scale (YY′ and XX′) so that they shall be at right angles, with their 0-points together, as shown in the figure. This is quickly done with cross-lined paper. Pupils should have some pages of cross-lined paper in their note-books.



Graph of $3x+5$

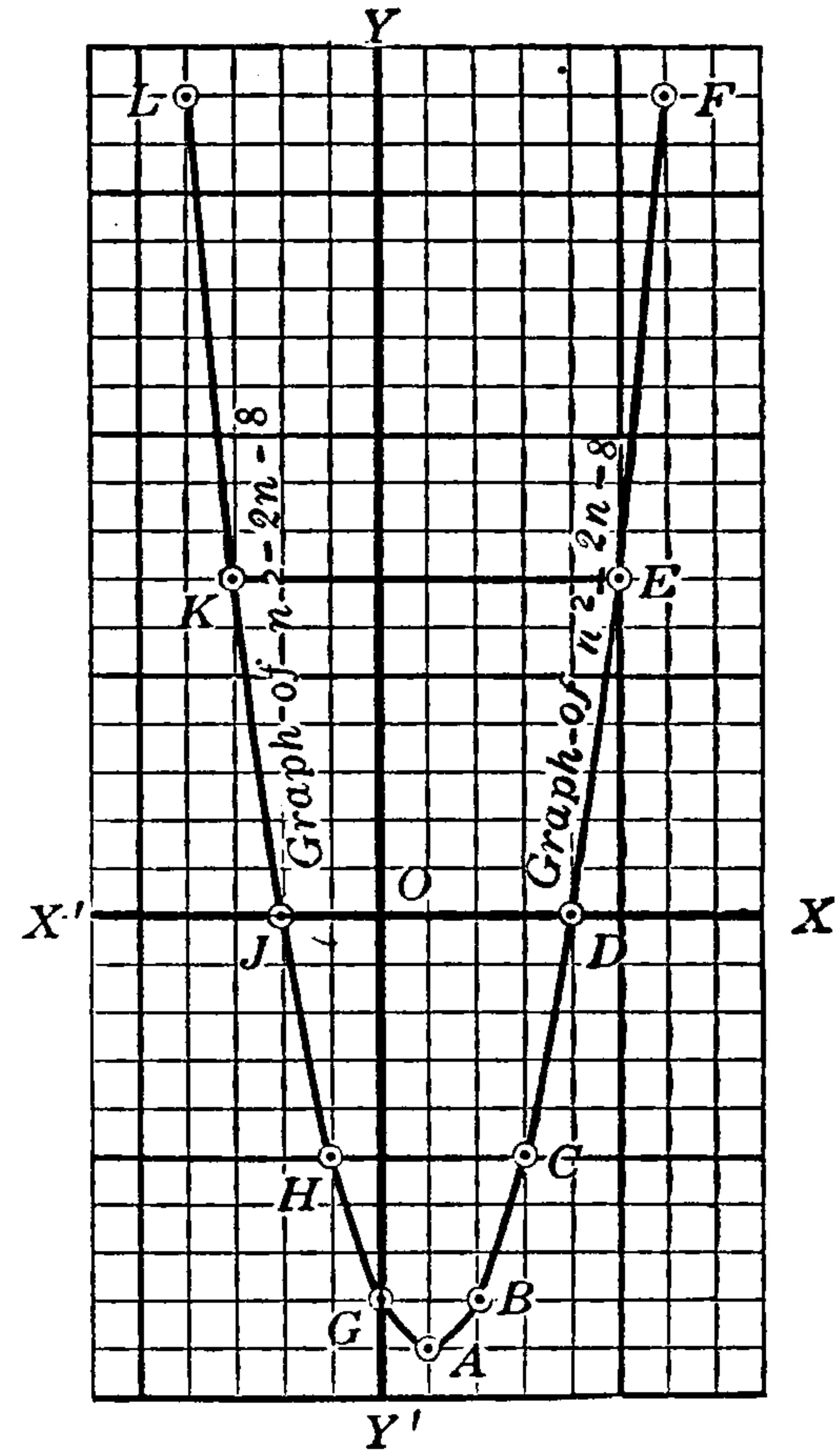
If you should substitute any whole or fractional positive or negative value in $3x+5$ for x and locate the point-picture of the resulting number-pair, you would always find that the point falls on this same line.

Try $x=\frac{1}{2}$, $1\frac{1}{2}$, $-\frac{1}{2}$, $-2\frac{1}{2}$, etc.

The conclusion is that $3x+5$ connects numbers into *number-pairs*, whose picturing points all lie along the *same straight line*.

Any number of pairs of values are given by $3x+5$.

What we have been doing in this section is called **graphing the function 3x+5**.



Graph of n^2-2n-8

80. Picturing n^2-2n-8. Let us now make a picture of n^2-2n-8.

Assume	$n=$	1,	2,	3,	4,	5,	6,	0,	-1,	-2,	-3,	-4,
then calculate												
	$n^2-2n-8=$	-9,	-8,	-5,	0,	$+7$,	$+16$,	-8,	-5,	0,	$+7$,	$+16$.

The number pairs are here $(1, -9)$, $(2, -8)$, $(3, -5)$, $(4, 0)$, $(5, 7)$, $(6, 16)$, $(0, -8)$, $(-1, -5)$, $(-2, 0)$, $(-3, +7)$, and $(-4, +16)$, the n-value being the first number of each pair.

Using again a pair of perpendicular algebraic scales on cross-lined paper, picture the number-pairs as in the figure.

Draw carefully freehand a smooth curve through points A, B, C, and so on to F and then to L, as shown.

In this case the number-pairs lie along a *curve*, called a *parabola*. The parabola is an *open* curve.

Any value you might take for n, substituted in n^2-2n-8, would give a number-pair whose point-picture would lie on this same curve.

Try $n=\frac{1}{2}$, $n=-\frac{1}{2}$, $n=1\frac{1}{2}$, $n=-1\frac{1}{2}$, etc.

The function n^2-2n-8, is then a number-law which pictures into a **parabola.**

What we have just been doing in this section is called **graphing** $\mathbf{f(n)=n^2-2n-8}$.

81. To make pictures of functions we merely *assume* values for x, or n, etc., substitute the assumed values in the functions ($3x+5$ or n^2-2n-8), and calculate the second numbers of the number-pairs. It then remains to picture the number-pairs on a pair of perpendicular algebraic scales, as above.

Any number of number-pairs are given by either $3x+5$ or n^2-2n-8, or by any other function.

Every such function has some straight or curved line-picture. The particular number-pairs given by any function always picture into points all of which lie on the same straight or curved line. Hence, every function has its own particular line-picture.

The rising and falling of the line or curve picture the changes in the function that are produced by changing the independent number, as x or n.

SOLVING EQUATIONS IN ONE UNKNOWN GRAPHICALLY

82. Solving $\mathbf{3x+5=8}$, Graphically. Suppose now that we were required to solve the equation $3x+5=8$.

We would calculate some number-pairs of $3x+5$, locate the picturing points (see figure in § 79), and draw through the points the straight line.

So soon as we know the line-picture to be a straight line, two rather widely separated points are sufficient to give the line-picture.

Since we want to find the value of x that makes

$$3x+5=8,$$

we measure 8 units up on the vertical scale, and draw a horizontal out until it crosses the line of $3x+5$. The length of this line, or its equal measured along the horizontal scale, is the required value of x. The length is 1, and as it extends to the *right*, $x=+1$. This horizontal is called the **graphical solution** of $3x+5=8$.

Notice that while any number of number-pairs are given by $3x+5$, only *one* of these number-pairs will make $3x+5=8$.

83. Solving $n^2-2n-8=7$, Graphically. Similarly, let it be required to solve $n^2-2n-8=7$, graphically.

Calculate some number-pairs by substituting values of n as in § 80, and draw the parabola-picture, freehand, as in § 80.

Since we are seeking the value of n that makes

$$n^2-2n-8=7,$$

we draw a horizontal through a point 7 units up on the vertical scale, and prolong the horizontal *both ways* until it crosses the parabola. The line is KE in the figure of § 80. It will cut the parabola in *two* points. The lengths of the parts of the horizontal between the vertical scale and the curve are the *two* values of n that will make

$$n^2-2n-8=7.$$

The two values are $n=+5$, and $n=-3$.

Substitute each of the two values in n^2-2n-8 and see if they make it equal 7. This shows that there are two values that will give the one value 7 for the algebraic number, n^2-2n-8.

Notice then that while any number of number-pairs are given by n^2-2n-8, only *two* of these pairs make

$$n^2-2n-8=7.$$

This means there are only *two* points on the graph of n^2-2n-8 where $n^2-2n-8=7$. They are the points K and E in the figure of § 80.

84. We have now shown how to make pictures of number-laws such as $3x+5$ and n^2-2n-8, and have also shown how to solve graphically such equations as $3x+5=8$ and $n^2-2n-8=7$. For any other algebraic numbers or equations that contain *only one letter*, the method is the same.

Exercise 26

Draw the line-pictures of the following functions of x:

1. $2x+5$
2. $x+5$
3. $x+3$
4. $2x+3$
5. $3x+2$
6. $3x+1$
7. $3x-1$
8. $2x-1$
9. $x^2+8x+12$
10. $x^2-3x-10$
11. x^2-2x-3
12. x^2-1
13. x^2-6x+8
14. x^2-6x+5
15. x^2-4x

Exercise 27

Solve the following equations graphically:

1. $2x+5=7$
2. $x+5=9$
3. $x+3=5$
4. $2x+3=-1$
5. $3x+2=8$
6. $3x+1=7$
7. $3x-1=5$
8. $2x-1=-5$
9. $x^2+8x+12=21$
10. $x^2-3x-10=0$
11. $x^2-2x-3=5$
12. $x^2-1=8$

SUMMARY

85. The work of this chapter has taught the following facts:

1. Algebraic numbers, or functions, require us to keep in mind two numbers, the function itself and also some other number, as x or n, that it depends on for its value.

2. An algebraic number or function is a shorthand description of the way to calculate its own value.

3. Algebraic numbers associate numbers into number-pairs.

4. The point-pictures of the number-pairs of an algebraic number give the line-pictures of the algebraic numbers, called the graphs of the algebraic numbers.

5. To find the value of an algebraic function when the value of the number it depends on is given, we substitute the given value and simplify.

6. To find the value of the independent number when the algebraic function is given equal to a number, we must solve an equation.

7. An equation is only a shorthand way of saying a function is to have a certain value.

8. While an algebraic function may furnish a great number of number-pairs, usually only one or a few of these pairs furnish a solution of the equation which gives the algebraic function a *particular* value.

Although the graphical solutions of equations make the meaning of solutions clear and comprehensible, even in minute details, still they are more tedious and cumbersome than the algebraic solutions. When it is only the *results* of solutions that are wanted, and after it is learned that algebraic solutions are shorter and easier ways of reaching these results, we shall use algebraic solutions.

Algebraic solutions are treated in the next chapter.

CHAPTER VI

EQUATIONS. GENERAL REVIEW

EQUATIONS

86. The equation is the backbone of algebra. Its value consists in its power as a tool for solving problems. Other algebraic topics are needed to give insight into and power over the equation. Algebraic skill means and always has meant nearly the same as skill in using the equation. In mathematical history the evolution of the equation means the evolution of algebra.

The earliest algebraists were the Egyptians. Thirty-five hundred years ago they said such things as, "A quantity, its half and third make 19. Find the quantity." They used no symbols or abbreviations, but the language of words only. About sixteen hundred years ago Diophantus, a Greek mathematician, wrote down the initial letters of the verbal sentence as his equation. It was simply a shortened sentence.

A thousand years later calculators wrote down rules for calculating *in symbols*, much as a postal clerk of our day might write down rules for calculating the postage on parcels for various zones. For example, if for zone 3 the postal rule is "6¢ for the first pound or fraction and 2¢ for each additional pound in the weight of the package," the postal clerk might write $2x+4$, in which x is the weight in pounds, as a short form of the rule. On weighing the package he might do as $2x+4$ says, *i.e.*, double the number of pounds and add 4 to get the number of cents to charge as postage.

Now if at the other end of the route the persons receiving the package had no scales and desired to know the weight of the package, knowing the postage to be 12¢, they might write down $2x+4=12$, and find what x is, if they could solve the equation.

Again, if a man starts 5 miles from his home and walks away from it x miles an hour for 2 hours, the rule for finding his distance from home would be $2x+5$. Suppose he did not know his rate but did know how far he was from home, say 13 miles. To find his rate he might write $2x+5=13$ and, if he knew how to solve the equation, he could find his rate, x, of walking.

At a later date men came to regard such forms as $2x+4$ and $2x+5$, not as shortened rules, but as the results of following the rules, *i.e.*, as *numbers*. Then they began to apply the laws of number to them, that is they began learning how to add, subtract, multiply, and divide them, and algebra was a reality.

87. Equations expressed partly or wholly in letters are either *identities* or *conditional equations*.

88. An **identity** is an equation with like members, or members which may be reduced to the same form.

89. The **sign of identity** is $\equiv$. It is read, *is identical with*, or *is identically equal to*, or simply *is*.

The sign of equality may also be used in an identity when there is no need to distinguish the nature of the equality.

Thus, $5a+4a+2a\equiv 8a+3a$, and $ax+c\equiv c+ax$ are identities, and it is evident that they are true for any value of each letter in them.

90. Substitution is the process of putting one number symbol into an expression in place of another *which has the same value*.

91. Satisfying an Equation. An equation is said to be *satisfied* by any number which, when substituted in place of the unknown number, reduces the equation to an identity.

The equation, $5x+3x=72$, is satisfied by $x=9$, for the substitution of 9 for x gives the identity, $45+27\equiv 72$.

Exercise 29

Perform the indicated operations and answer the questions in the following:

1. $(3a+6b)\times 2$ $\quad$ $(3n-10)\times 3$ $\quad$ $(8x-20y)\div 4$

2. If $x+8$ is the present age of a man, how old is another man who is twice as old?

3. $(5x-12)\times 3$ $\quad$ $(4a-8b)\div 4$ $\quad$ $(6x+120)\div 6$

4. If Tom has x dollars and Frank $3x-20$, how many has Fred who has half as many as both the others?

5. $(8x-9y)\times 4$ $\quad$ $(2n+15)\times 4$ $\quad$ $(5a-35b)\div 5$

6. If x is one number and $2x-10$ another, what is a third number which is twice the sum of the other two?

94. In the statement of many problems, one or both members may contain a known and an unknown number. Thus,

$$7x-4=8+5x$$

Before solving, it is necessary to have all unknown numbers in one member and all known numbers in the other member.

If by the addition axiom, § 15, we add $+4$ and $-5x$ to both members of the equation without uniting similar terms, we have

$$7x-5x=8+4$$

The same result might have been obtained by subtracting $+5x$ and -4 from both members of the equation.

95. This process of changing a term from one member of an equation to the other without destroying the equality is called **transposition**.

To avoid mechanical work and to impress upon themselves what axiom is involved in this change, students should always explain the work by telling what they *add to* or *subtract from* both members.

Exercise 30

In like manner solve and check the following equations, applying the addition and the subtraction axioms alternately:

1. $5x-32=3x-16$
2. $14-4n=n+32-8n$
3. $13-6s=25-9s$
4. $9b+12=6b+40-b$
5. $8y+14=4y+74$
6. $15-3x=x+75-9x$
7. $9n-19=44+2n$
8. $3s-s-18=36-8s$
9. $32-2x=72-6x$
10. $7a+6-15=79-4a$
11. $6b+16=3b+26$
12. $10+9n=88+2n-8$
13. $34-5b=49-8b$
14. $6x-14=56-2x+2$
15. $9s-13=4s+27$
16. $16+4n+7=3n+30$
17. $23-3x=71-7x$
18. $4a-15-a=35-2a$

Exercise 31 — Oral Practice

Do this entire list of 14 exercises in 26 minutes.

1. A has x sheep, and B has y. How many would C have, if he had twice as many as A and B?

2. Indicate by use of parentheses the product of the sum and difference of any two numbers, as m and n.

3. If there are x hundreds, y tens, and z units in a number, what will represent the number?

4. What will represent the sum of four consecutive odd numbers of which n is the largest?

5. How many square feet are there in the walls of a room x feet square and n feet high?

6. The sum of the ages of 4 men is $10x$ years. What was the sum of their ages 12 years ago?

7. If n represents an integer, does $2n+2$ represent an even or an odd number? Show why.

8. From x dollars a man paid two debts, one of a dollars and the other of b dollars. How much did he have left?

9. A paid x dollars for a harness and $4x$ dollars for a horse. Represent the cost of both.

10. If one part of x is 16, what is the other part? If one part of y is 45, what is the other part?

11. A boy bought x oranges at m cents apiece and sold them at n cents apiece. If he lost, what was his loss?

12. The difference between two numbers is 25, and the smaller number is s. What is the larger number?

13. Represent the number of acres in a rectangle of land x rods long and $x-5$ rods wide.

14. A house cost n dollars, a farm $5n$ dollars, and a store $4n$ dollars. Express in two ways the cost of all.

Exercise 32 — Review Problems and Equations

Solve and check the following problems and equations:

1. The sum of two numbers is 128, and their difference is 34. Find the larger number.

2. $7x-13=x+12+5$

3. $6s+17=45-2s+8$

4. Divide the number 184 into two parts so that the greater shall exceed the less by 48.

5. $9n-80=26-n+4$

6. $3y+12=16-5y+4$

7. The sum of two numbers is 270, and their difference is 4 times the smaller. Find the numbers.

8. $18+3x=40-x+7$

9. $7b-50=23-2b-1$

10. A and B own a farm worth \$13,100. A has 3 times as large a share as B. How much is B's share?

11. $4n-15+n=5-5n$ **12.** $60-3s=6s-8s+7$

13. One automobile ran 3 times as fast as a second and 6 miles an hour faster than a third. The sum of their rates was 120. Find the rate of the third.

14. $16+5x-38=7-x$ **15.** $8a+30=35+7a-3$

16. Three times a number diminished by 57, is equal to twice the number increased by 68. Find the number.

17. $8y-40=50-y+6$ **18.** $9n-15=37+2n+4$

19. A horse and carriage cost \$385, the horse costing \$95 more than the carriage. What did the horse cost?

20. A and B are 57 miles apart. They travel toward each other until they meet, A traveling twice as many miles as B. How many miles did A travel?

21. A has twice as many acres of land as B, and B has three times as many acres as C. If all of them have 2400 acres, how many acres have A and B together?

Exercise 33 — Oral Practice

Do this entire list in 28 minutes.

1. A merchant sold x yards of silk for \$45. What will represent the cost per yard?

2. If a man has a half-dollars and b quarters, how many cents has he? How many dollars?

3. Indicate the sum of a and b, diminished by c. The sum of $3x$ and x, diminished by y.

4. What will represent the sum of three consecutive numbers of which s is the smallest? Of which s is the middle number?

5. If there are x tens and y units in a number, what will represent the number?

6. How much butter, at b cents a pound, will pay for n pounds of tea at 60 cents a pound?

7. What will denote the number of square feet in a piece of paper l yards long and w feet wide?

8. A farmer received x dollars for sheep which he sold at y dollars a head. How many did he sell?

9. Find the value of a bushels of apples at m cents a peck and b bushels of pears at n cents a peck.

10. If a represents an integer, when does $a+1$ represent an even number? When an odd number?

11. If the difference between two numbers is 45 and the larger one is x, what is the smaller number?

12. What will represent the sum of three consecutive even numbers of which s is the smallest? s the largest?

13. The sum of two numbers is 175, and the difference between them is 5 times the smaller. Find the numbers.

14. The sum of the ages of 3 boys is $6x$ years. If they live, what will be the sum of their ages in 8 years?

CLEARING EQUATIONS OF FRACTIONS

96. Clearing of Fractions. An equation containing fractions must be changed so as to remove the fractions before it can be solved. Observe that

$$\tfrac{4}{5}\times 20=16$$

Multiplying this fraction by 20, a *multiple* of its denominator, the product is a whole number. Multiplying any fraction by a *multiple* of its denominator gives a whole number, for the denominator cancels with one factor of the multiplier.

97. Principle.— *If any fraction is multiplied by a multiple of its denominator, the product is a whole number.*

98. Problem.— To clear of fractions, the equation

$$\frac{x}{2}-10+\frac{x}{4}+3=\frac{x}{3}-5+\frac{x}{6} \qquad (1)$$

Multiply both members of this equation by 12, the least **common multiple** of the denominators, by multiplying each term in it, applying cancellation to the fractional terms, and the result is

$$6x-120+3x+36=4x-60+2x \quad \text{(Mult. Axiom)} \qquad (2)$$

Every term in this equation is a whole number. This work is called **clearing an equation of fractions.**

In describing this transformation of an equation, students should tell by what they multiply both members of the equation, rather than use the expression, *clearing of fractions, i.e.,* they should say: "by the use of the multiplication axiom," etc.

Solving equation (2), $x=8$

Checking in (1): $\frac{8}{2}-10+\frac{8}{4}+3=\frac{8}{3}-5+\frac{8}{6}$

$4-10+2+3=\frac{24}{6}-5$

or, $-1=-1$

Exercise 34

Clear of fractions, solve, and check the following:

1. $\frac{x}{2}-4+\frac{x}{3}=3+\frac{x}{4}$

2. $\frac{s}{3}-12+\frac{2s}{15}=8-\frac{s}{5}$

3. $\frac{n}{5}+2+\frac{n}{3}=4+\frac{n}{2}$

4. $\frac{5y}{12}+y-\frac{y}{8}=38+\frac{y}{2}$

5. $\frac{x}{2}-3+\frac{x}{9}=2+\frac{x}{3}$

6. $\frac{2a}{15}+a-\frac{a}{2}=18+\frac{a}{3}$

7. $\frac{s}{3}+8-\frac{s}{8}=6+\frac{s}{4}$

8. $\frac{3n}{25}+n-\frac{n}{5}=44+\frac{n}{3}$

Exercise 35 — Problems and Equations

Solve and check the following:

1. A woman bought silk at \$2 a yard and had \$14 left. Twice as many yards at \$1.50 a yard would have cost \$4 more than she had. Find the cost of the silk bought.

Let n = the number of yards she bought;
then $2n+14$ = the number of dollars she had,
and $3n-4$ = the number of dollars she had.

$$3n-4=2n+14$$

$n=18$, and $2n=36$ (\$36, cost of silk.)

Check:
$$3\cdot 18-4=2\cdot 18+14$$
$$50=50$$

2. A has twice as many sheep as B and 35 less than C. If all have 635, how many has A?

3. $\frac{x}{4}+6-\frac{x}{3}=7-\frac{x}{9}$

4. $\frac{2s}{3}+2\frac{1}{2}+\frac{s}{2}=s+\frac{3s+5}{10}$

5. A boy has $\frac{1}{4}$ as many 5-cent pieces as dimes. If he has \$9 in all, how many coins has he?

6. $\frac{y}{3}-5+\frac{y}{9}=\frac{y}{5}+6$

7. $\frac{n}{4}-1\frac{1}{4}+n=\frac{6n-9}{3}-\frac{2n}{3}$

8. A and B together earn \$200 a month; A and C, \$215; B and C, \$235. How much do all earn?

9. $\frac{x}{3}+8-\frac{x}{5}=\frac{x}{4}-6$

10. $\frac{s}{3}+\frac{4s+5}{7}-\frac{2s}{10}=s-6\frac{2}{3}$

11. Two horses cost \$350, one costing $1\frac{1}{2}$ times as much as the other. Find the cost of each.

12. Half of a number, diminished by 6, is equal to $\frac{1}{3}$ of the number, increased by 2. Find the number.

13. The sum of the ages of mother and daughter is 48 years, and the difference between their ages is four times the daughter's age. Find the mother's age.

GENERAL REVIEW

Exercise 36 — Oral Review

Do this page in 15 minutes.

1. Express six times the product of a and b, increased by 3 times the sum of x and y.

2. What will represent the sum of 4 consecutive numbers of which x is the largest?

3. A man's capital doubled for 3 successive years, when it was $16,800. How much had he at first?

4. What is the age of a man who y years ago was a times the age of a boy whose age was x years?

5. How many square yards are there in the walls of a room $3x$ feet by $2x$ feet and y feet high?

6. What will represent the sum of 5 consecutive numbers of which m is the middle one?

7. A boy had a dollars. He earned b dollars and then spent c dollars. How much did he have left?

8. If one number is n and another number is 4 times as large, what is the sum of the numbers?

9. A farm cost 3 times as much as a house. If the farm cost $6200 more than the house, what did both cost?

10. If a field is x rods square, how many rods of fence will be required to enclose it and divide it into 4 squares?

11. A girl has x quarters, y dimes, and z nickels. Give an expression to denote how many dollars she has.

12. What will denote the number of feet in the perimeter of a rectangle $5x$ feet long and $3x$ feet wide?

13. A man bought x sheep at a dollars a head and had b dollars left. How much money had he at first?

14. A house cost 3 times as much as the lot, one costing $5000 less than the other. What did both cost?

Exercise 37 — Written Review

Solve all the problems of this page in 20 minutes.

1. A's age is to B's as 5 to 7, and the sum of their ages is 132 years. Find the age of each.

Let $5n$ = the number of years in A's age,
and $7n$ = the number of years in B's age.

$$5n + 7n = 132$$

The pupil will understand that the number sought is not the value of n, but the numbers represented by $5n$ and $7n$.

2. B's age is to A's as 4 to 7, and the difference between their ages is 27 years. Find A's age.

3. Seven boys and 12 men earn $275 a week. If each man earns 4 times as much as each boy, how much do the 7 boys earn per week?

4. A has 3 times as many cows as B; but if A should sell 6 to B, they would then have the same number. How many cows have both men?

5. Three men engage in business with a capital of $11,000. B invests half as much as A and $200 more than C. How much have A and B invested?

6. A, B, C, and D have 290 sheep. B has 15 more than A, C has 15 more than B, and D has 15 more than C. How many have A and B?

7. Three men raised 1684 bushels of oats. A raised 3 times as many bushels as C, and 185 bushels more than B. How many bushels did B and C raise?

8. A horse, carriage, and harness cost $350. The horse cost $95 more than the harness, and the carriage $35 less than the horse. Find the cost of the horse.

9. A boy bought oranges at 3¢ apiece and had 20¢ left. At 5¢ apiece, he would have needed 16¢ more to pay for them. How many did he buy?

Exercise 38 — Questions and Problems

1. Define *algebraic expression*; *term*; *monomial*; *polynomial*; *similar terms*; *value of an algebraic expression.*

2. From what expression must $9x+6y-5z$ be subtracted to give $-4x-3y+5z$?

3. How are $8\cdot 9$ and $6\cdot 9$ added in arithmetic? Why cannot $9a$ and $8a$ be added in the same manner?

4. What are the factors of a number? Distinguish between the *parts* of a number and the *factors* of it.

5. How may numbers which are expressed by 2 factors be added, if they have a common factor?

6. How is $4\cdot 8$ subtracted from $9\cdot 8$ in arithmetic? In what other way might it be subtracted?

7. What expression must be subtracted from $7a-5b+4c$ to give $9a+6b-2c$?

8. Define *identity*; *equation of condition.* Give examples and show how they differ.

9. Simplify $9a-(2b-c)+2d-(5a+3b)+4c-2d$, and find its value if $a=8$, $b=-4$, $c=-5$.

10. Add $a(a+b)+2(b+c)+2(b-c)$, $-3(b+c)+2(a+b)+b(b-c)$, and $a(b+c)-a(a+b)-4(b-c)$.

11. How is the correctness of subtraction proved in arithmetic? Is the same test applicable in algebra?

12. Subtract $2z+x-2u+y+7$ from the sum of $4x-2y+5z-u$ and $3y+6-4z-2x$.

13. Perform two different operations on an equation so that one term shall be transposed from each member.

14. Describe four operations which change the *form* and *value* of the members of an equation, but not their *equality.*

15. What expression must be added to $6x-5y+4z$ to give $9x+4y-7z$?

16. In the identity, $5x+3x-2x\equiv 10x-4x$, what number does x represent?

17. From $3b-2c+5d-4e$ subtract the sum of $3d-5e-4c+2b$ and $c+e+2d-4b$.

18. How do you subtract one term from another, if the two terms are partly similar?

19. To what expression must $8a-4b+9c$ be added to give $5a+2b-6c$? To give 0?

20. Define *root* of an equation. How do you determine whether a number is a root of an equation?

21. Subtract $2a-4\text{b}+5$ from 0, and add the difference to the sum of $5a-3c$ and unity.

22. Name the different steps in the solution of a problem by the use of an equation. Illustrate.

23. What must be true of two number expressions in order that we may place them equal to form an equation?

24. State the principle for enclosing two or more terms of a polynomial in a parenthesis. Illustrate.

25. How are terms that are partly similar added? Write 3 terms that are partly similar and add them.

26. From the sum of $2xy+3xz-yz$ and $3z-2xz+xy$ subtract the sum of $xz-yz$ and $5z+3xy-xz$.

27. Add $a(a-x)-2(a+x)+a(a-2)$, $3(a+x)+(a+3)-(a-2)-a(a-x)$, and $a(a+3)-(a+x)$.

28. How do you prove whether the numbers found in solving a problem satisfy the conditions of the problem?

CHAPTER VII

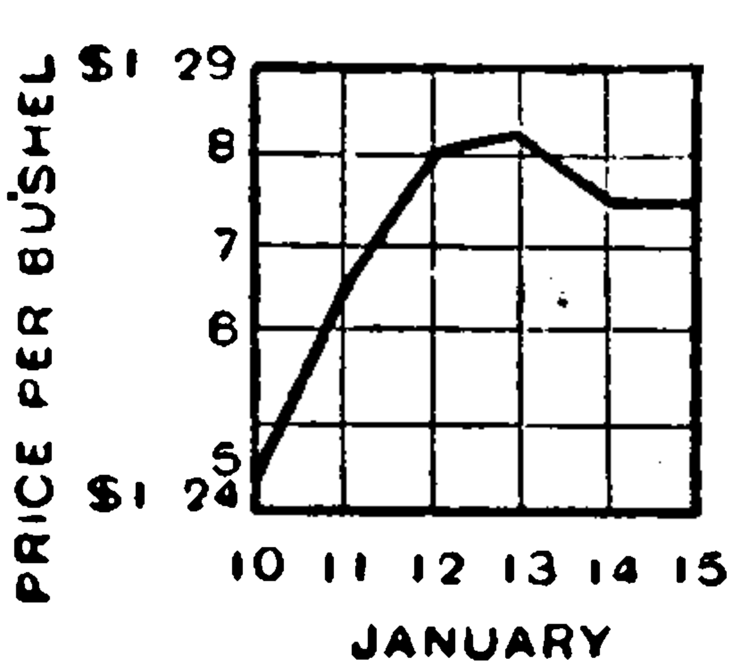
GRAPHING DATA. SOLVING SIMULTANEOUS EQUATIONS GRAPHICALLY

GRAPHING DATA

99. Graphing, as was illustrated in Chapter V, means representing by pictures and diagrams.

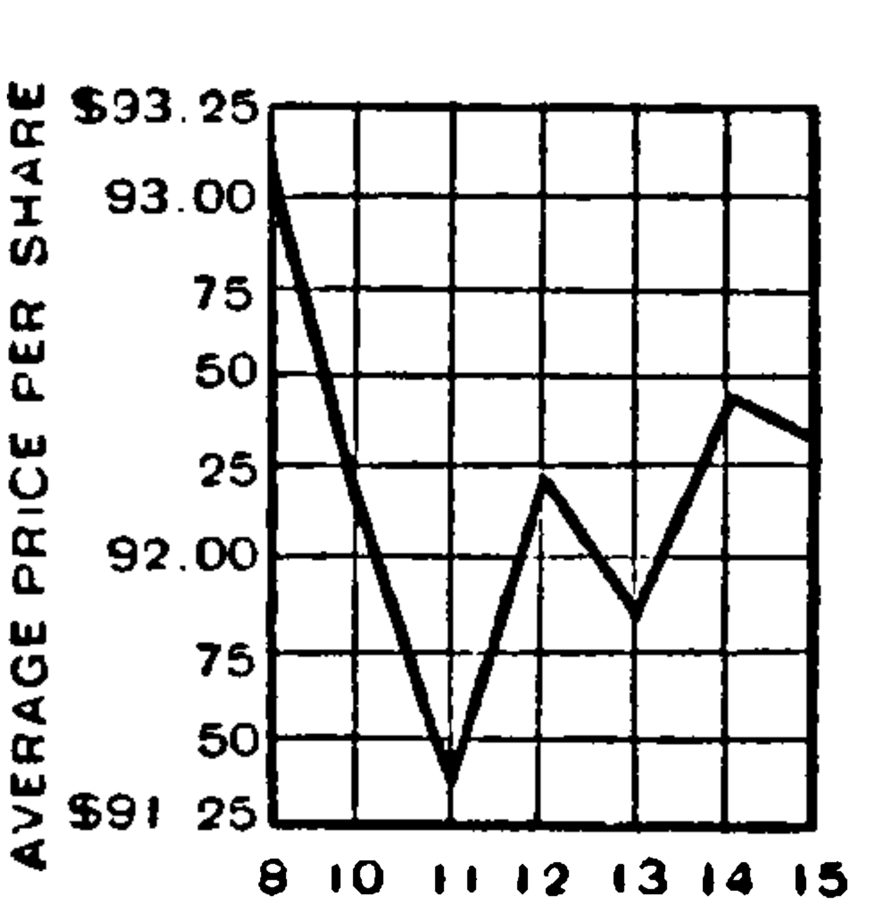
100. The diagrams and exercises below show how to picture laws that connect two sets of related numbers, such as prices and dates, temperatures and times, etc., when the laws cannot be expressed as equations, as well as when they can be so expressed.

Exercise 39



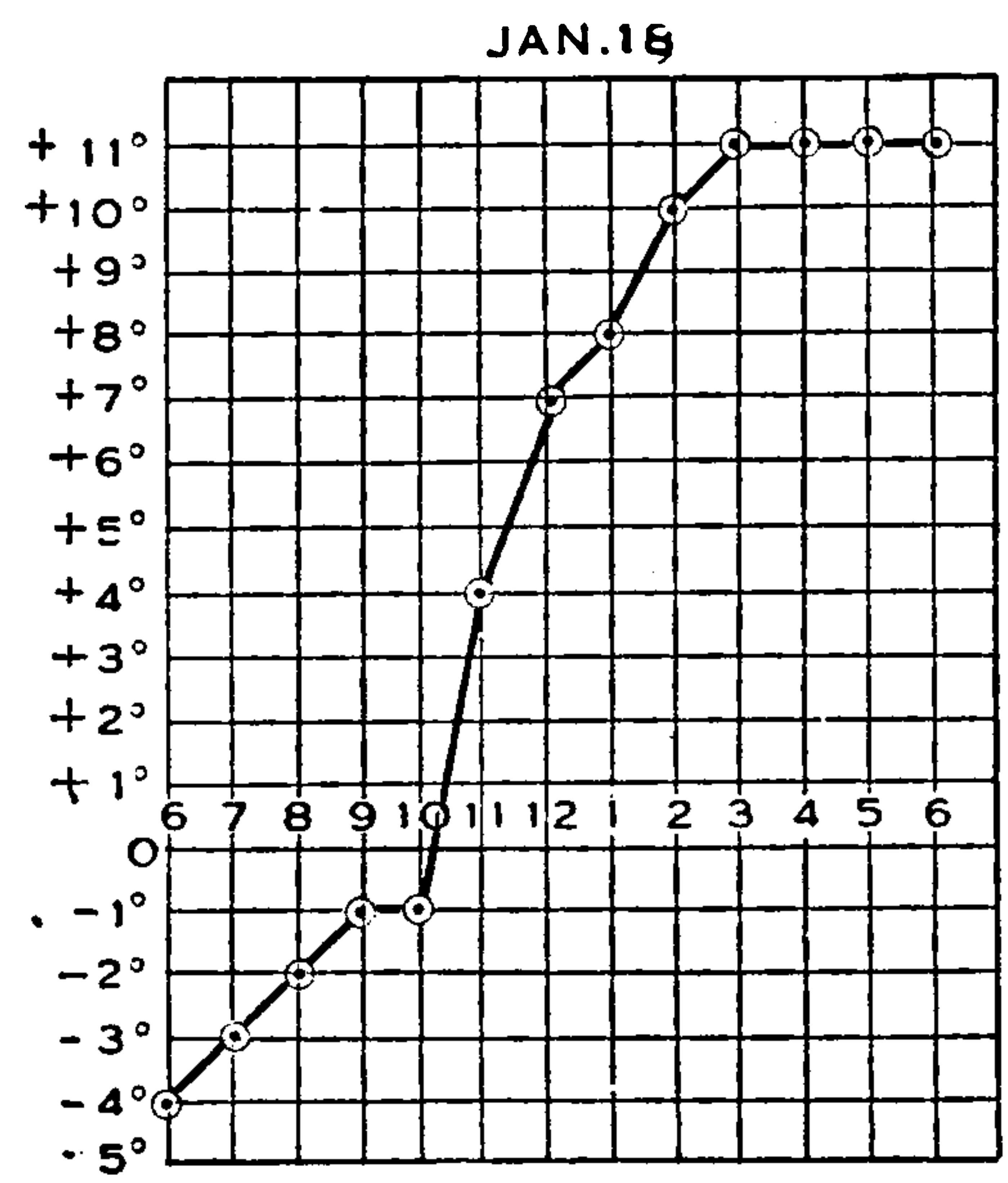
1. In a newspaper of January, 1916, the prices of wheat from Jan. 10 to 15 on a Board of Trade were given as in the figure. The numbers along the horizontal are the dates, and those along the vertical, the prices per bushel. What was the price of wheat on Jan. 10? On Jan. 11? 12? 13? 14? 15?

2. On what date was the price highest? Lowest? Between what dates did the price change most?



3. The average price per share, for dates Jan. 8-15, 1916, of 20 leading stocks of the New York Stock Exchange, was as shown in the figure. How much did the price fall from Jan. 8 to Jan. 10? Between what other dates did the price fall? Rise? What day was the rise greatest? The fall greatest?

4. What was the average price of these stocks on Jan. 11? On Jan. 14? On Jan. 15?



5. The hourly temperatures from 6 a. m. to 6 p. m. of Jan. 18, 1916, in Chicago, were as shown in the figure. Observe the degree-numbers along the vertical and the hour-numbers along the horizontal, and give the temperature at 6 a. m.; at 9 a. m.; at 12 m.; at 2 p. m.; at 6 p. m.

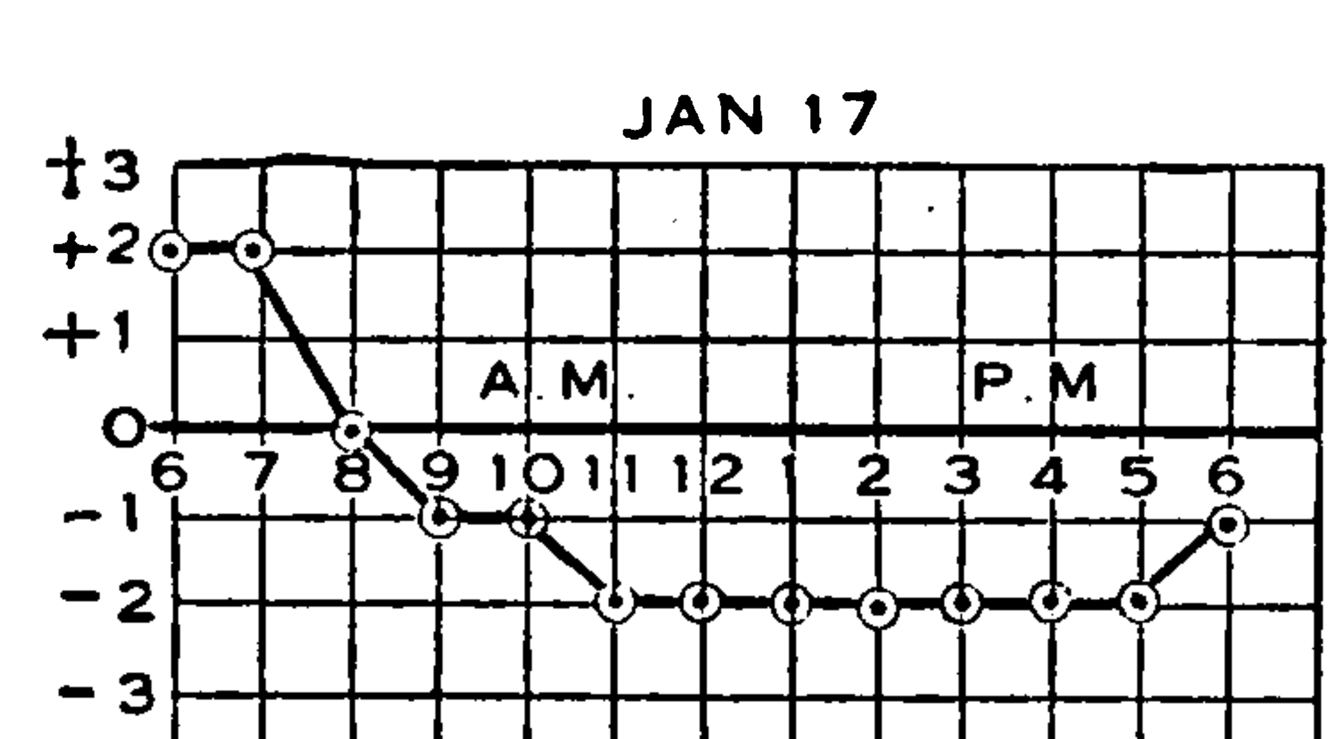
6. At what hour was the temperature lowest on Jan. 18? At what hours highest? When does the graph show the temperature stationary?

In graphing temperatures, the lines connecting the points that represent hourly readings do not represent the temperatures for the intermediate points. The temperature was probably not stationary at any time. But from the hourly readings it was apparently stationary. Nevertheless, the graphs give a good notion of the general trend of the temperature for the day.

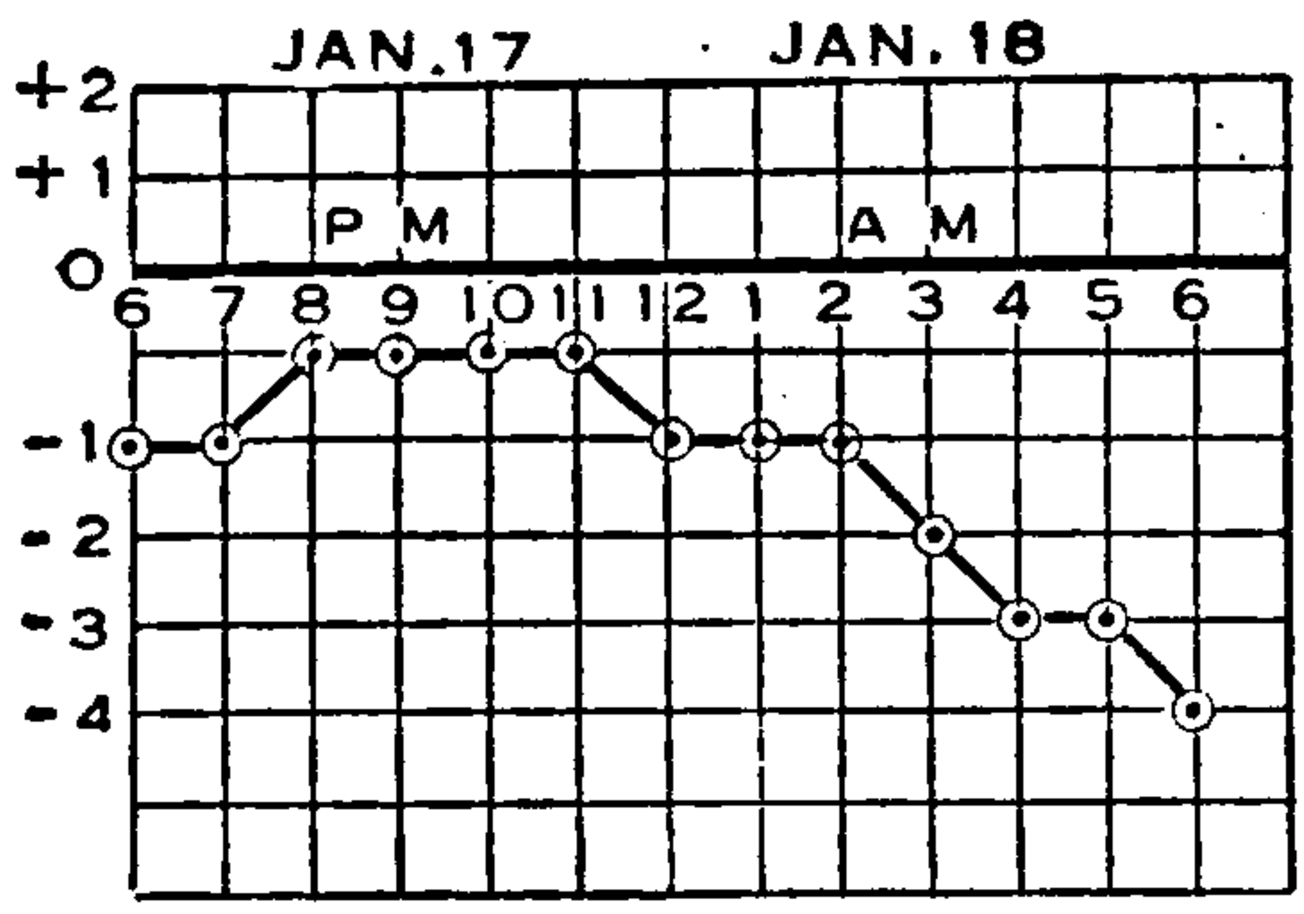
7. The hourly thermometer readings from 6 a. m. to 6 p. m. on Jan. 17, 1916, in Chicago, were:

	A. M.						M.	P. M.					
Hours	6,	7,	8,	9,	10,	11,	12,	1,	2,	3,	4,	5,	6
Reading	+2°,	2°,	0°,	−1°,	−1°,	−2°,	−2°,	−2°,	−2°,	−2°,	−2°,	−2°,	−1°

Show that the temperature line is as given in the figure. When was it coldest? Warmest? When growing colder? Warmer? When stationary by the graph?

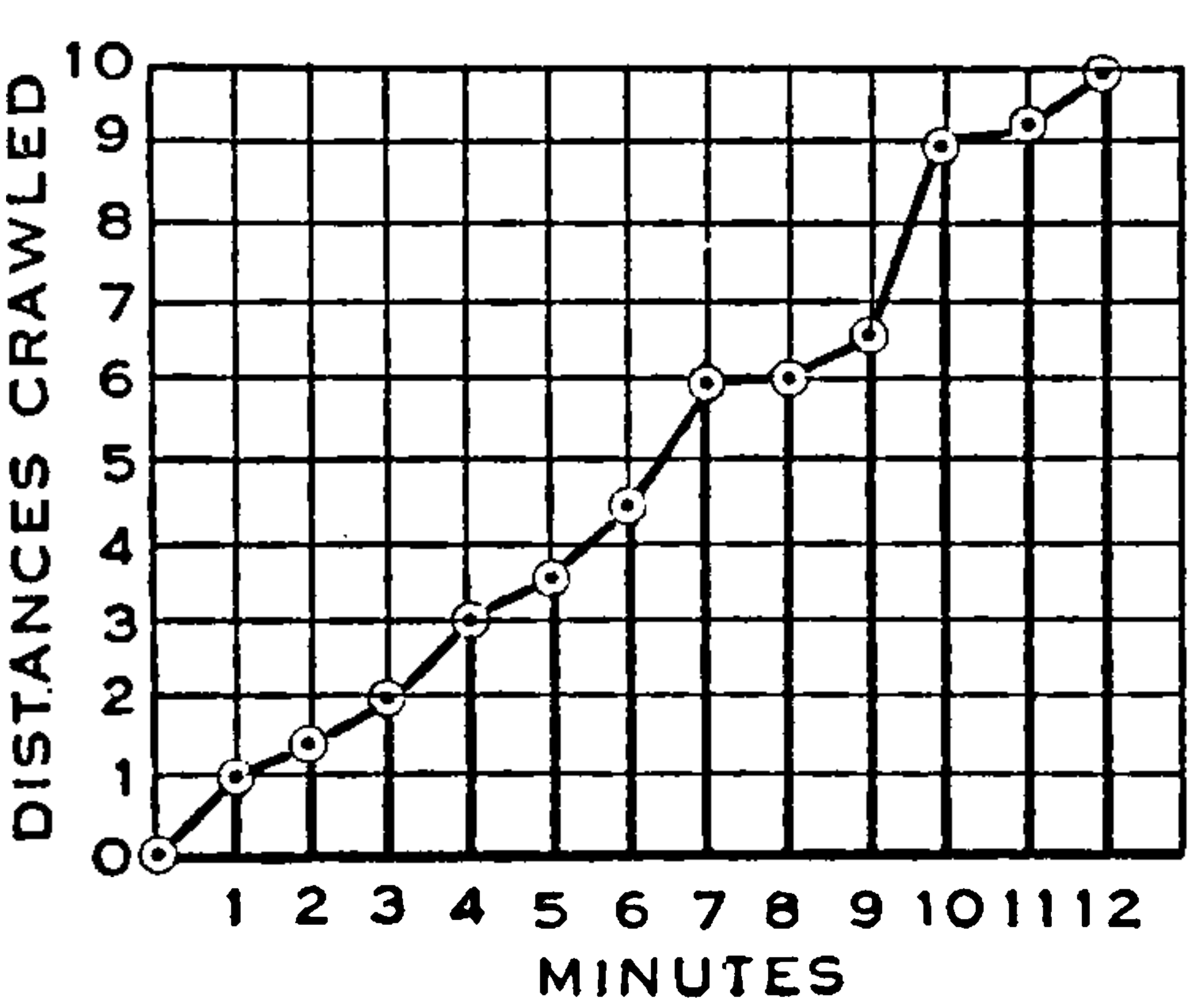


See note after problem 6.



8. The hourly temperature curve from 6 p. m. Jan. 17 to 6 a. m. Jan. 18, 1916, was as shown in the figure. What was the thermometer reading at 7 p. m.? At 8, 9, 10, and 11 p. m.? At midnight? At 3 a. m. of the 18th? At 5 a. m.? At 6 a. m.?

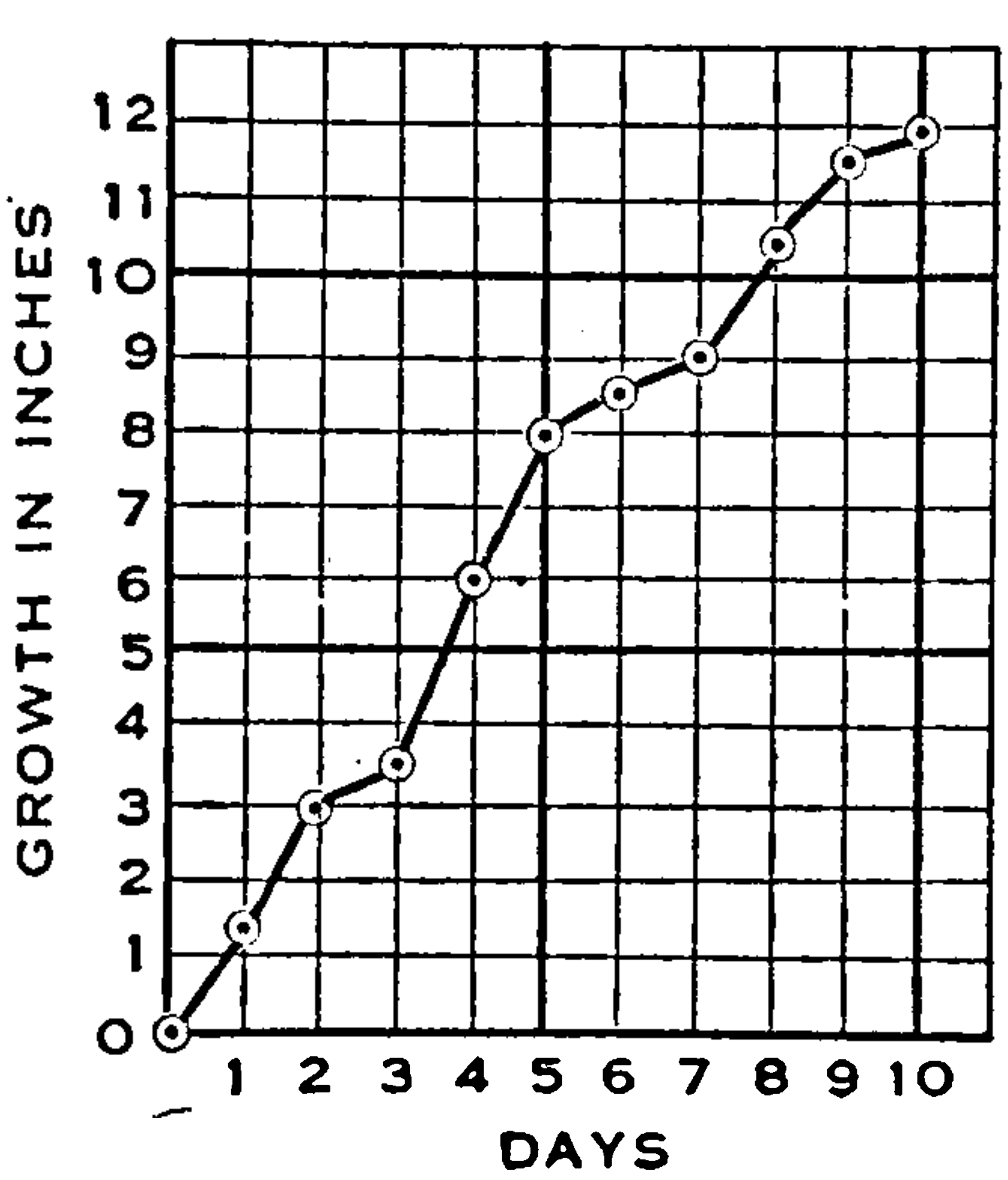
9. From Jan. 17, 6 p. m. to Jan. 18, 6 a. m. when was it growing warmer? Colder? When stationary by the graph?



10. A class studied the movement of a snail by having it crawl along a foot-rule. The observing time, in minutes, was written along the horizontal, and the distances crawled, in inches, along a vertical, giving a picture of the snail's rate of crawling as in the figure. How far had the snail crawled the first minute? The first 2 min.? In 4 min.? In 6 min.? In 12 min.? What minute did it crawl most rapidly? Most slowly?

11. The daily growths of a tulip in inches were:

Day........	0	1	2	3	4	5	6	7	8	9	10
Height.....	0	$1\frac{1}{4}$	3	$3\frac{1}{2}$	6	8	$8\frac{1}{2}$	9	$10\frac{1}{2}$	$11\frac{1}{2}$	12

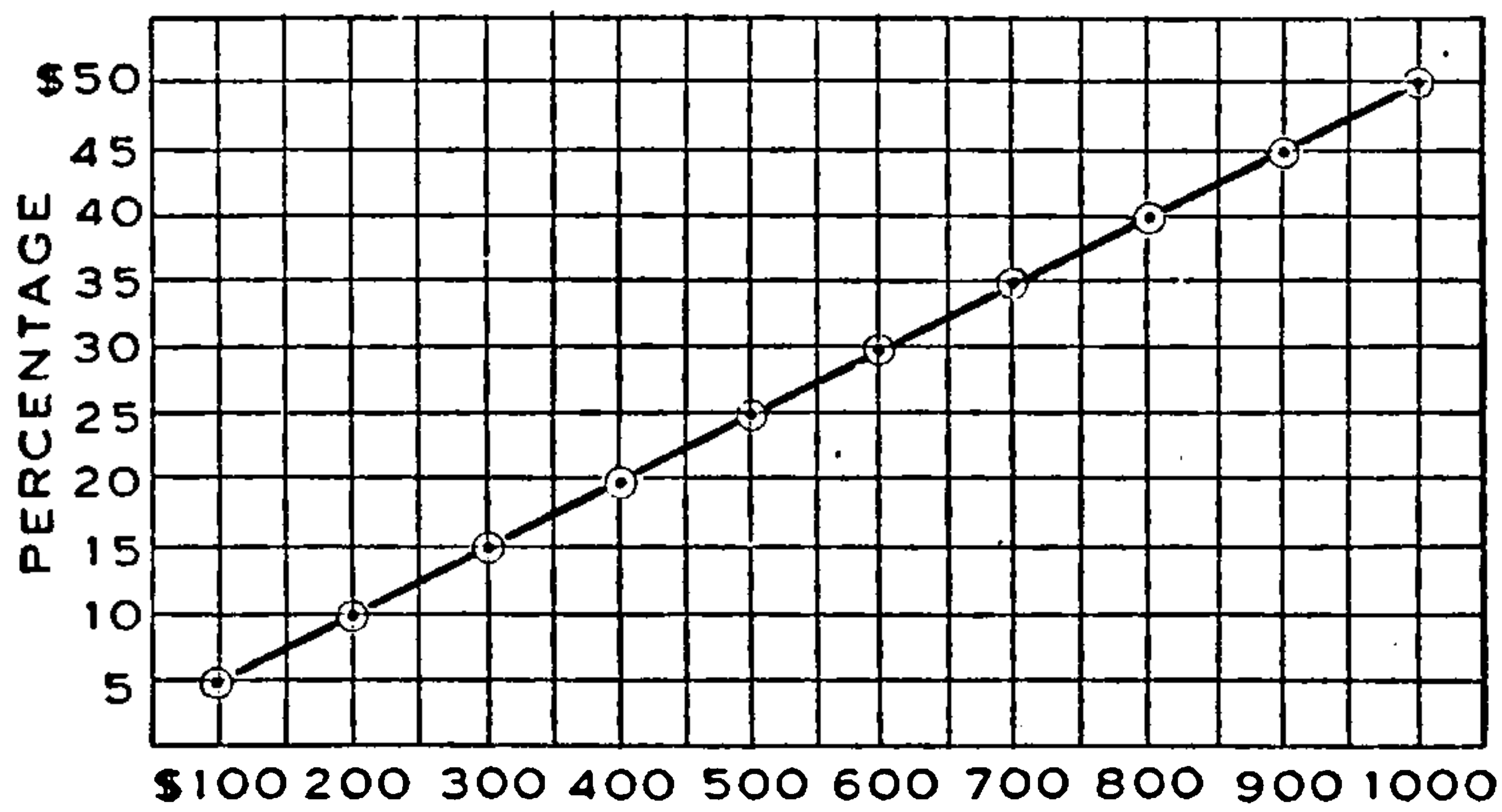


Mark off the days along a horizontal and the growths along verticals through 1, 2, 3, etc., using a scale of 1 short side to 1 inch, and draw a broken line connecting the points.

What was the least growth on any day? The greatest growth?

17. The rate per cent being 5, graph the following percentages and bases:

Base.......	\$100,	\$200,	\$300,	\$400,	\$500,	\$600,	\$700,	\$800,	\$900,	\$1000
Percentage..	\$5,	\$10,	\$15,	\$20,	\$25,	\$30,	\$35,	\$40,	\$45,	\$50



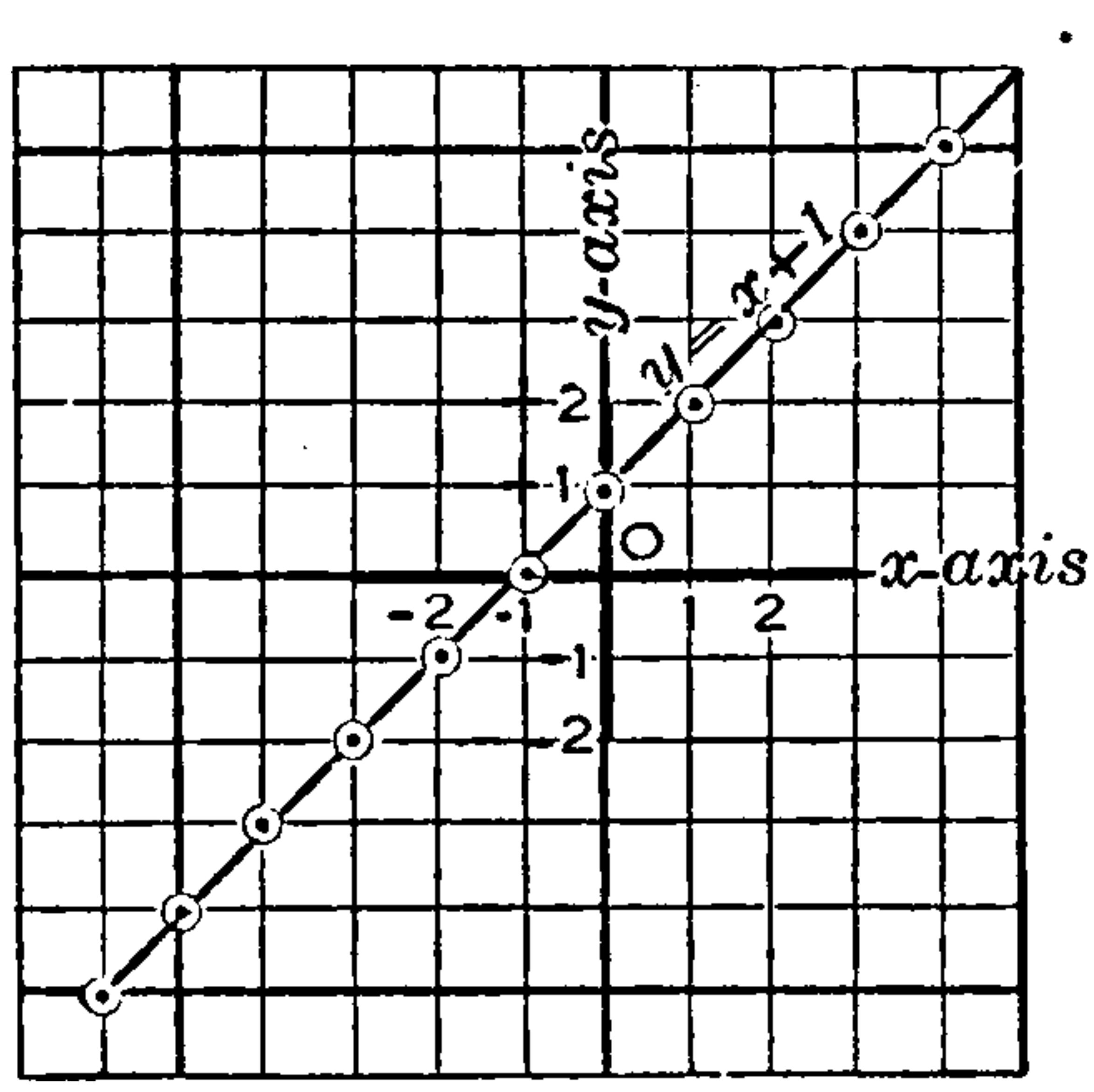
Base Scale
1 horizontal space = \$50
1 vertical space = \$5

There is also an algebraic expression of this law, thus,

$$p = 5 \cdot \frac{b}{100} \text{ or } p = \frac{b}{20}$$

18. If now we take an algebraic law like $y = x + 1$, we may substitute successive values of x, right and left from 0, and *calculate* the corresponding values of y, thus:

$x = 1,$	$2,$	$3,$	$4,$	$5,$	$0,$	$-1,$	$-2,$	$-3,$	$-4,$	$-5,$	etc.
$y = 2,$	$3,$	$4,$	$5,$	$6,$	$1,$	$0,$	$-1,$	$-2,$	$-3,$	$-4,$	etc.

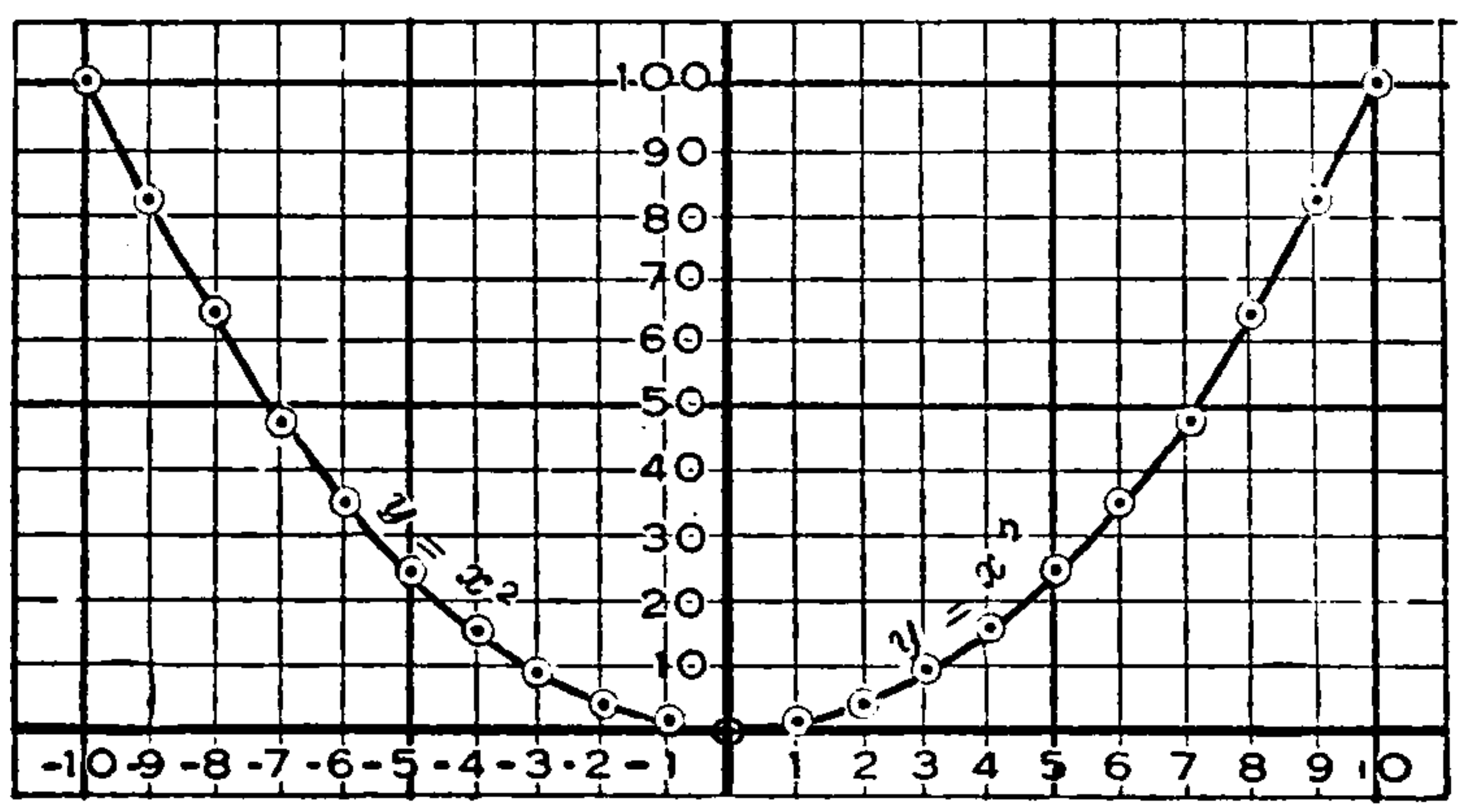


Graph of $y = x + 1$

Mark off the x-values along the horizontal, to the right if positive, and to the left if negative. Measure, to a convenient scale, the corresponding y-values on the verticals, upward if positive and downward if negative. Connect the points with a line. This line is the graph of $y = x + 1$.

19. Graph the algebraic law $y=x^2$, by substituting successive values for x and calculating in $y=x^2$, the corresponding values of y.

$x=0$,	1,	2,	3,	4,	5,	-1,	-2,	-3,	-4,	-5,	etc.
$y=0$,	1,	4,	9,	16,	25,	1,	4,	9,	16,	25,	etc.



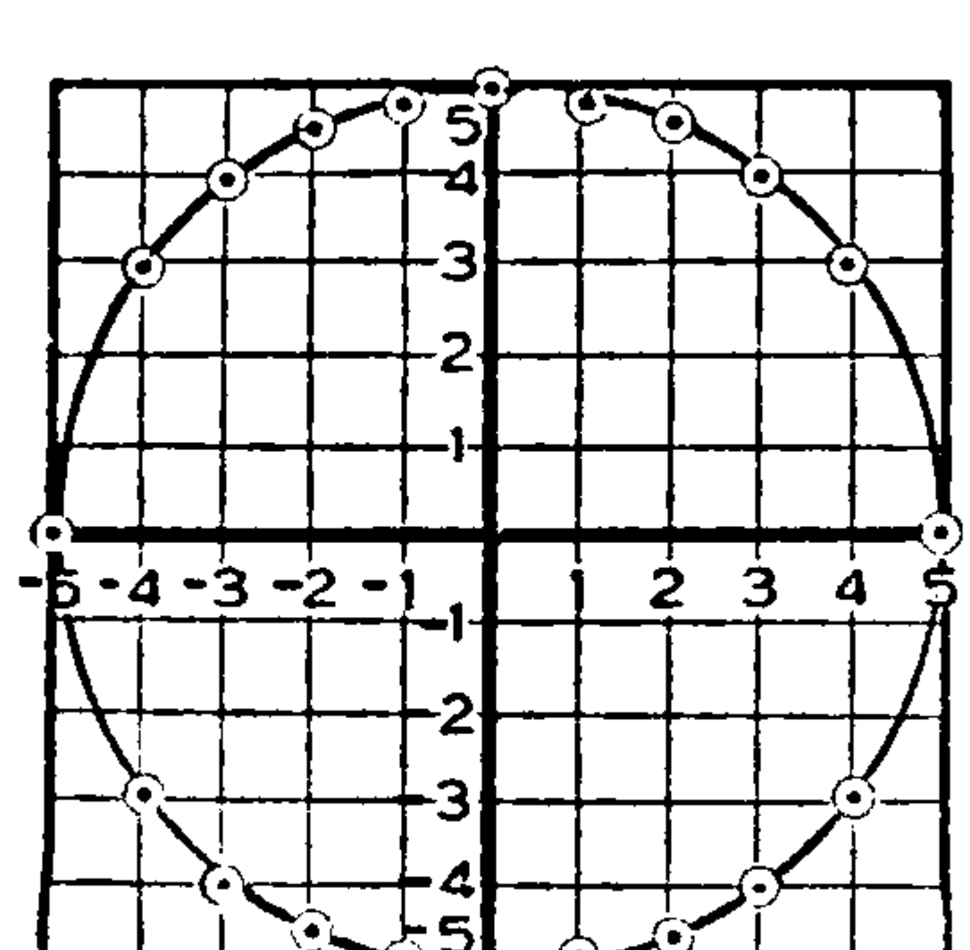
Graph of $y=x^2$

Scale
1 horizontal space = 1
1 vertical space = 10

Mark the x- and y-values off on horizontal and verticals.

20. Graph the algebraic law $x^2+y^2=25$, or $y= \pm\sqrt{25-x^2}$,* by calculating values of y and plotting points as above:

$x=$	0	1	2	3	4	5	-1	-2	-3	-4	-5, etc.
$y=$	± 5	$\pm 4.9+$	$\pm 4.6+$	± 4	± 3	0	± 4.9	± 4.6	± 4	± 3	0, etc.



Graph of $y= \pm\sqrt{25-x^2}$
or $x^2+y^2=25$

*The expression $\sqrt{25-x^2}$ means the square root of $25-x^2$. The sign $\pm$ means that the number calculated for $\sqrt{25-x^2}$ may be either positive or negative.

101. From the above problems it is seen that a group of facts expressed by two different sets of connected numbers, like dates and prices, times and temperatures, ages and heights, x-values and y-values in an equation, may be pictured, or graphed. This is generally done by measuring off the numbers of one set horizontally and of the other set vertically, locating points, and then connecting the points.

102. Problems 18, 19, and 20 have shown the following important facts:

1. A single equation in two unknowns is satisfied by *many pairs* of values of the unknowns.

2. By measuring off x-values horizontally and y-values vertically to suitable scales, locating points and connecting them, equations may give either straight or curved line graphs, or pictures.

3. Every pair of values of x and y that satisfies a given equation gives a point-picture that lies on one and the same line or curve.

4. It is easy to see that the x- and y-distance of any point on the curve from the chosen reference lines, would, if substituted, *satisfy* the equation that gave the graph.

103. In problem 18 the graph of $y=x+1$ was found to be a *straight line.* This could be shown by stretching a string along the row of points. *Any equation in two unknowns in which each unknown has the exponent** 1 (as $3x-2y=1$) *gives a straight-line graph.* Knowing this, it is easy to draw graphs of such equations by merely choosing *two* values of x, calculating the corresponding *two* values for y, locating the two points, and drawing a straight line through the two points with a ruler.

*With numbers like x, x^2, y, y^2, the small number written (or understood) at the right and above the letter is called an **exponent.** When no number is written, as with x, or n, or y, 1 is understood to be the exponent, just as though the written forms were, x^1, or n^1, or y^1.

A third point may well be calculated and located as a *check on the work.*

It is best not to take the values of x too near together, as it is difficult to draw a line accurately through two very near points.

104. Linear Equations. Since equations in two unknowns both with exponent 1, have straight-line graphs, they are commonly called **linear equations.**

1. Graph the linear equation

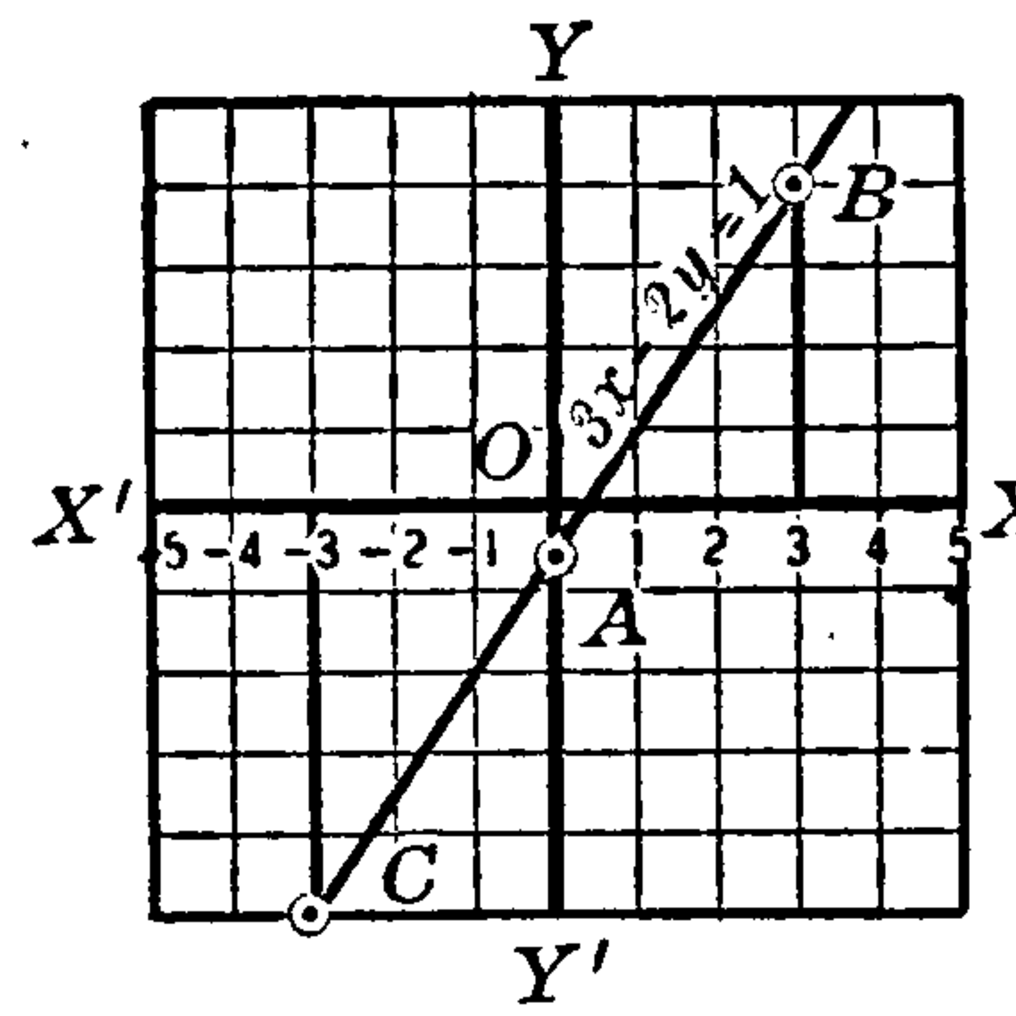
$$3x-2y=1.$$

Take $\quad x=0, \quad +3, \quad -2,$ and
compute $\quad y=-\frac{1}{2}, \quad +4, \quad -3\frac{1}{2}.$

The number-pairs for the points are written thus:

$$(0, -\tfrac{1}{2}), (3, 4), (-2, -3\tfrac{1}{2}),$$

the first number in the parenthesis being the x-value.



Linear Equation
Straight-Line Graph
Graph of $3x-2y=1$

Graph the first two points $(0, -\frac{1}{2})$ and $(3, 4)$, as at A and B, draw a line through them with a ruler, and test whether the point $(-2, -3\frac{1}{2})$ lies on the line, as at C.

2. In a similar way graph each of the following equations:

1. $y=x-2$	2. $y=x-4$	3. $y=2x$
4. $y=4-x$	5. $y=2x-1$	6. $y=2x+3$
7. $x+2y=6$	8. $2x-y=4$	9. $3x-4y=4$

105. We have just seen that one linear equation in two unknowns is satisfied by *many* pairs of values of x and y. But *two* linear equations in two unknowns, such as

$$2x+y=7$$
$$2y-x=4$$

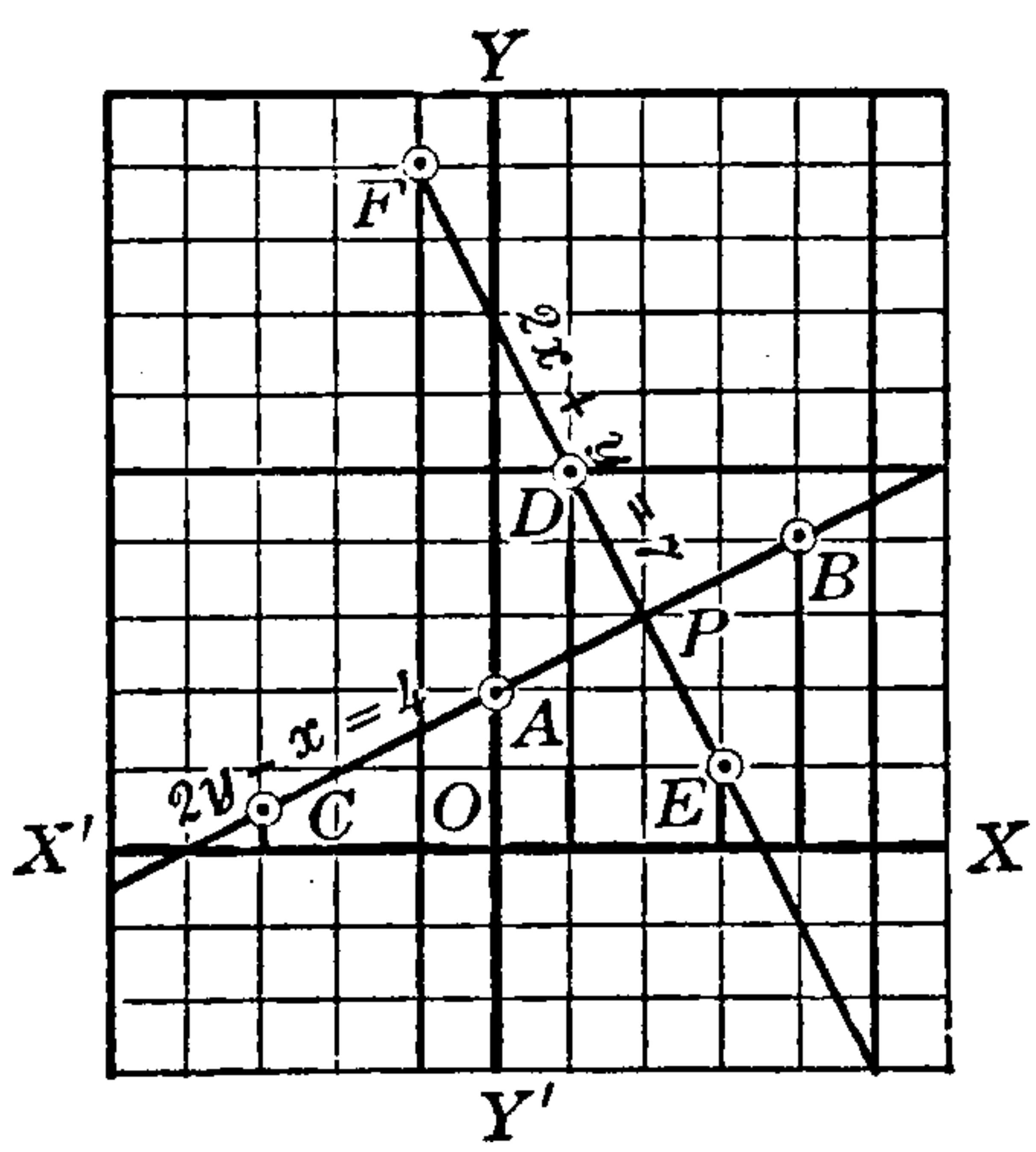
can both be satisfied at the same time by *only one pair* of values of x and y.

For example, graph $2x+y=7$,

using $x=+1$, $+3$, and -1,
giving $y=+5$, $+1$, and $+9$,

and graph $2y-x=4$,

using $x=\ \ 0$, $+4$, and -3,
giving $y=+2$, $+4$, and $+\frac{1}{2}$. (See figure.)



Simultaneous Equations
Intersecting Graphs

Now, we ask, can a point lie so as to give x- and y-distances that will satisfy *both* equations?

The answer is *yes*. The point of intersection, P, of the graphs satisfies the requirement. For the point, P, $x=+2$ and $y=+3$, and these values satisfy both equations. Hence, the x- and y-distances of the point of intersection of the graphs are **the graphical solution** of the two given linear equations. Since the graphs cross at only *one* point there is *only one* solution of the pair of equations.

106. Hence, *two linear equations in two unknowns can be satisfied by only* **one** *pair of values of the unknowns.*

SOLVING SIMULTANEOUS EQUATIONS GRAPHICALLY

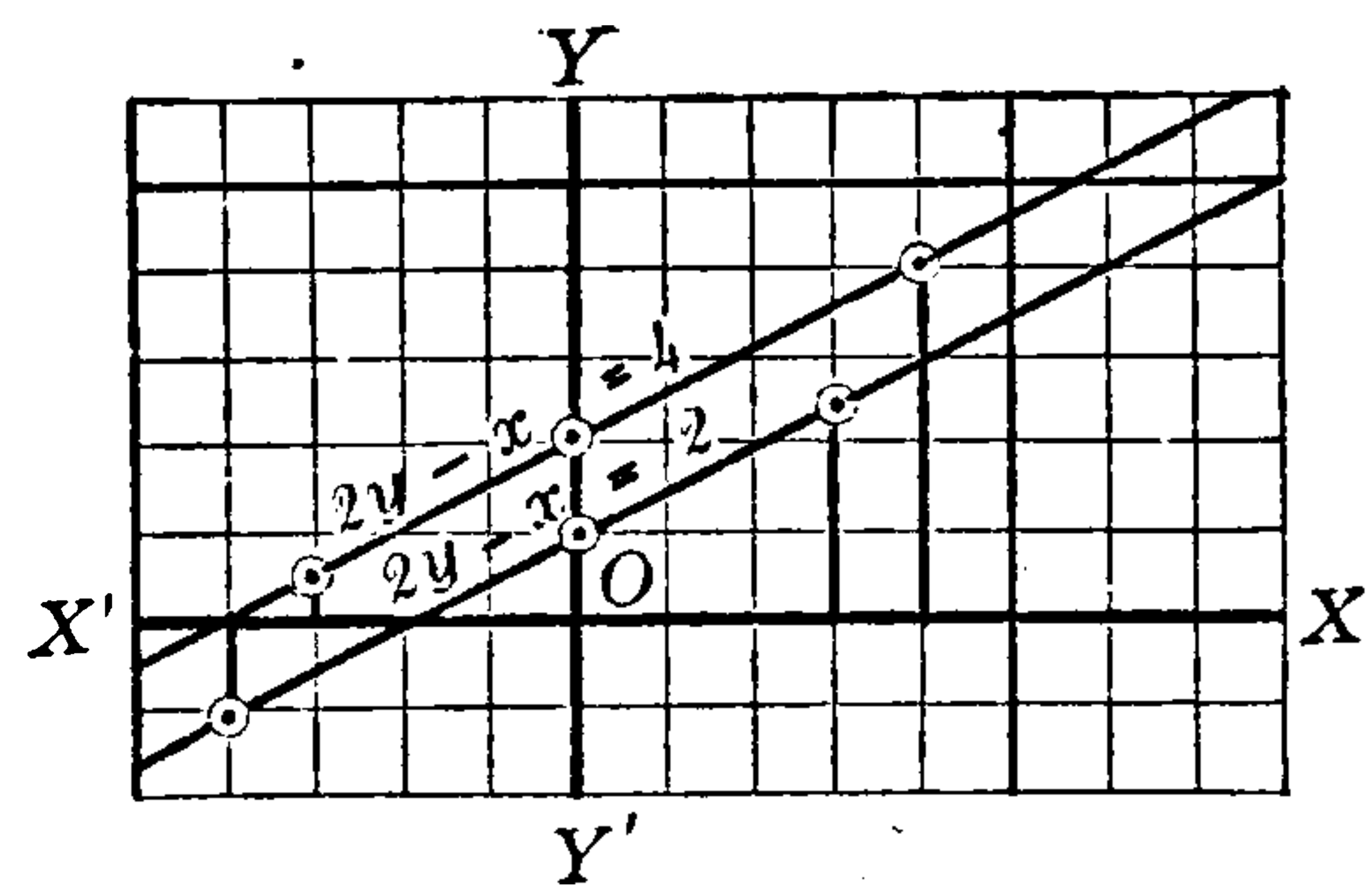
107. Simultaneous Equations. Equations that can be satisfied by the same values of the unknowns are called **simultaneous equations.**

108. It is now worth while to see that *not all* pairs of linear equations in two unknowns can be satisfied by even one pair of values of the unknowns.

Two or more equations considered together are said to form a **system.**

1. Consider the system $\begin{cases} 1. & 2y-x=4 \\ 2. & 6y-3x=6 \end{cases}$

The graphs of the equations are shown in the figure. Dividing 2 through by 3, gives $2y-x=2$, and the graph on which this is written is the graph of $6y-3x=6$. The graphs are a pair of *parallel lines*. They do not meet, and there is *no* point that lies on both graphs. This means there is *no* pair of values of x and y that will satisfy both equations.



Non-Simultaneous Equations
Parallel Graphs

109. Inconsistent Equations. Equations which cannot be satisfied by any pair of values of the unknowns are called **non-simultaneous, or inconsistent equations.**

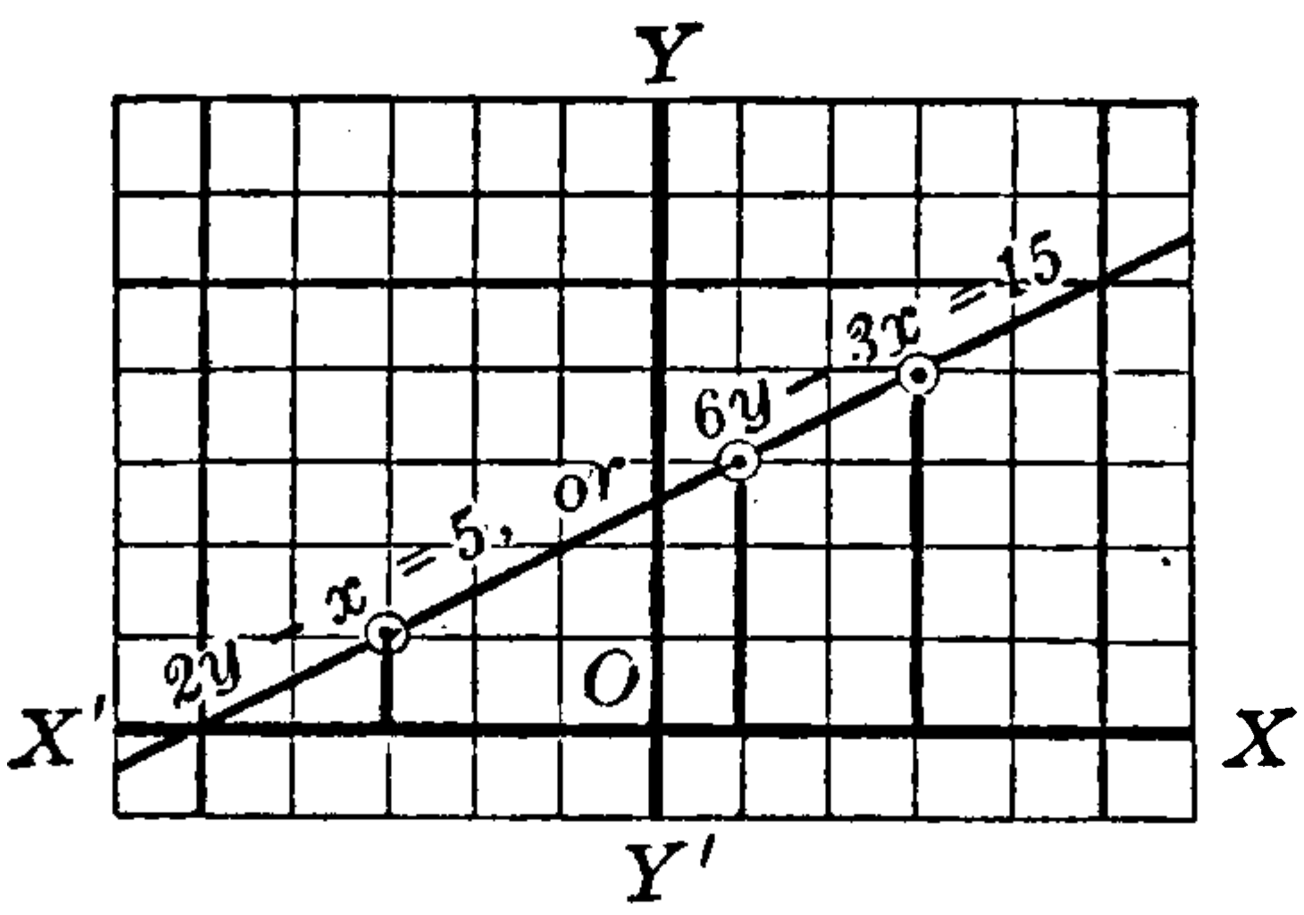
That the equations of § 108 are inconsistent can be seen without graphing, by dividing the second through by 3. This does not change the relation between x and y. Then one equation says that $2y-x=4$, and the other that $2y-x$ is *at the same time* equal to 2. This is obviously absurd. The number, $2y-x$, cannot *at the same time* be both 4 and 2.

110. *For a system of two linear equations in two unknowns to be capable of solution, the equations must be* **simultaneous.**

111. Dependent Equations. It is, however, not *sufficient* that the equations be simultaneous. We shall now see that two linear equations in two unknowns can fail to give a *definite* solution because they have *too many* solutions.

1. Consider the system, $\begin{cases} 1. & 2y-x=5 \\ 2. & 6y-3x=15 \end{cases}$

Both graphs are shown in the figure as a single line. They coincide. *Every* point that is on one is on the other also. Hence, *any* pair of values of x and y that satisfies one of the equations, satisfies the other also. Dividing the second equation through by 3, gives $2y-x=5$, which is *identical* with equation 1. One equation *depends* on the other in the sense that one can be derived from the other by simple division by an arithmetical number.



Dependent Equations
Coincident Graphs

Such equations are called **dependent equations.**

112. Finally, *for a system of two linear equations in two unknowns to be capable of solution, the equations must be both* ***simultaneous*** *and* ***independent.***

Exercise 40 — Graphical Solutions

Solve the following systems graphically, or in case there is *no* definite solution, tell whether the system is *inconsistent* or *dependent*:

1. $\begin{cases} x-y=2 \\ 3x-2y=9 \end{cases}$ 2. $\begin{cases} x+y=1 \\ 2x+5y=11 \end{cases}$ 3. $\begin{cases} x+y=2 \\ 3x+3y=6 \end{cases}$

4. $\begin{cases} x+y=5 \\ x-3y=1 \end{cases}$ 5. $\begin{cases} x+2y=6 \\ 2x+4y=12 \end{cases}$ 6. $\begin{cases} y=x-3 \\ 3x-5y=11 \end{cases}$

7. $\begin{cases} 2x-5y=15 \\ 5y-2x=-15 \end{cases}$ 8. $\begin{cases} y=2x-3 \\ x+2y=14 \end{cases}$ 9. $\begin{cases} 5x-3y=3 \\ 2x+y=10 \end{cases}$

The graphical way of solving equations makes the meaning of solutions clear; but the algebraic way of the next chapter is shorter, and as it can be applied to equations in 3, 4, 5, and even n unknowns, it is also much more generally useful than the graphical way.

116. Independent equations are equations which cannot be derived one from the other by addition of, or multiplication or division by a positive or negative arithmetical number.

The equations given above are independent, for one cannot be derived from the other by simple multiplication and division. So also are $4x+3y=28$ and $2x+3y=14$.

117. A system of equations is two or more equations involving two or more unknown numbers, as,

$$\begin{cases} x+2y=32 \\ x-2y=12 \end{cases} \quad \text{and} \quad \begin{cases} 2x+3y=36 \\ 6x-2y=20 \end{cases}$$

By a **set of roots** is meant the values of the unknown numbers in a system.

As has been noted, each equation of a system, when taken by itself, is indeterminate. It was noted, also, that only one set of roots will satisfy two independent equations. In the two systems above, $x=22$ and $y=5$ in the first and $x=6$ and $y=8$ in the second, were the sets of roots.

Simultaneous simple equations were solved graphically in Chapter VII. They will now be solved algebraically.

To solve two simultaneous equations containing two unknown numbers, it is necessary to obtain from them a *single equation* containing but *one unknown* number.

This can be done only in case the equations are **independent** as well as **simultaneous**; see § 112.

118. Elimination is the process of combining two or more simultaneous equations containing two or more unknown numbers in such a way as to obtain a single equation in which one of the unknown numbers does not appear.

ELIMINATION BY ADDITION OR SUBTRACTION

119. The following examples indicate the method of **elimination by addition and by subtraction.**

Solve the systems:

1.
$$\begin{cases} x+y=8 & (1) \\ x-y=6 & (2) \end{cases}$$
$$\begin{aligned} 2x &= 14 \\ x &= 7 \end{aligned}$$

We add (2) to (1), member to member, eliminating y, and then find the value of x.

2.
$$\begin{cases} 3x+3y=9 & (1) \\ 3x+\ y=5 & (2) \end{cases}$$
$$\begin{aligned} 2y &= 4 \\ y &= 2 \end{aligned}$$

We subtract (2) from (1) eliminating x, and then find the value of y.

We then substitute these values in one of the equations of the system that gave it, and find the value of the other unknown number.

From (1). $y=1$

checking $\begin{cases} x+y=7+1=8 & (1) \\ x-y=7-1=6 & (2) \end{cases}$

From (2). $x=1$

checking $\begin{cases} 3x+3y=3\cdot 1+3\cdot 2=9 & (1) \\ 3x+\ y=3\cdot 1+\ 2\ =5 & (2) \end{cases}$

In example 3, given below, we multiply both members of (2) by 2 and eliminate y by subtracting (3) from (1).

3.
$$\begin{aligned} 9x+4y&=43 && (1) \\ 3x+2y&=17 && (2) \\ \hline 9x+4y&=43 && (1) \\ 6x+4y&=34 && (3) \\ \hline 3x\ \ \ &=\ 9 \end{aligned}$$

4.
$$\begin{aligned} 3x+2y&=21 && (1) \\ 2x+3y&=19 && (2) \\ \hline 6x+4y&=42 && (3) \\ 6x+9y&=57 && (4) \\ \hline 5y&=15 \end{aligned}$$

In example 4, we multiply (1) by 2 and (2) by 3 and eliminate x by subtracting (3) from (4).

120. Rule.— *Determine first which of the two unknown numbers it is more convenient to eliminate.*

By the multiplication axiom, §15, make the coefficients of that unknown number the same in both equations.

If the signs of the terms to be eliminated are unlike, add the equations, member to member; if alike, subtract one equation from the other, member from member.

Exercise 41

Solve the following equations, checking some of them:

1. $\begin{cases} 2x+4y=8 \\ 3x-2y=4 \end{cases}$

2. $\begin{cases} 4x-2y=-8 \\ x+3y=-9 \end{cases}$

3. $\begin{cases} 4x-5y=1 \\ 2x-2y=2 \end{cases}$

4. $\begin{cases} 5x+3y=-4 \\ 2x+\ y=-1 \end{cases}$

5. $\begin{cases} 8x+6y=6 \\ 6x-3y=2 \end{cases}$

6. $\begin{cases} 2x+3y=-4 \\ 3x+5y=-5 \end{cases}$

7. $\begin{cases} 5x-3y=23 \\ 7x-4y=33 \end{cases}$

8. $\begin{cases} 3x-3y=-6 \\ 7x-6y=-1 \end{cases}$

9. $\begin{cases} 5x+3y=38 \\ 9y-3x=15 \end{cases}$

10. $\begin{cases} 6x+5y=-5 \\ 4y+5x=-7 \end{cases}$

11. $\begin{cases} 7x-3y=29 \\ 9x-4y=35 \end{cases}$

12. $\begin{cases} 5x+3y=-5 \\ 6y+9x=-6 \end{cases}$

13. $\begin{cases} 9x+8y=\ \ 12 \\ 4y-6x=-1 \end{cases}$

14. $\begin{cases} 4x+6y=-8 \\ 8y+5x=-4 \end{cases}$

PROBLEMS

121. Solving Problems. In algebra many problems in which two or more numbers are to be found can be solved by the use of a single equation containing but one unknown number, but in many problems it is more convenient to introduce as many unknown numbers as there are numbers to be found. Such solutions involve a *system of simultaneous equations*, and to make a solution possible, there must be *as many independent equations* as there are unknown numbers used.

9. A man sold 80 sheep for \$390, selling some of them at \$4 a head and the rest at \$6 a head. How many sheep did he sell at each price?

10. The sum of the ages of A and B is 92 years. If B were twice as old as he is, his age would exceed A's age by 16 years. Find the age of each.

11. In an election 5163 men voted for two candidates, and the candidate elected had a majority of 567. How many votes did each candidate receive?

12. The sum of two numbers is 255, and $\frac{2}{3}$ of the larger is equal to $\frac{3}{4}$ of the smaller. By how much does the larger number exceed the smaller?

13. Twelve men and 6 boys earn \$24 a day, and at the same daily wages, 7 men and 8 boys would earn \$16.25 a day. How much does each man earn per day?

14. A miller mixes corn worth 80¢ a bushel with oats worth 60¢, making a mixture of 100 bushels worth 72¢ a bushel. How many bushels of each does he use?

15. Eight years ago B was 3 times as old as A, but if both live 8 years, B will be only twice as old as A. What was the age of each 8 years ago?

16. A merchant sold 48 yards of silk for \$89, selling part of it at \$1.75 a yard and the rest at \$2 a yard. How many yards of the better silk did he sell?

17. A has 160 sheep in two fields. If he takes 15 from the first field to the second, he has the same number in each field. How many are there in each field?

CHAPTER IX

MULTIPLICATION

122. Multiplication is the process of taking one number as an addend a certain number of times.

$$3 \times 5 = 5 + 5 + 5 = 15$$

123. The **multiplicand** is the number taken as an addend.

124. The **multiplier** is the number which denotes how many times the multiplicand is taken.

125. The **product** is the result of multiplication.

THE SIGN OF THE PRODUCT

126. Taking $+5$ *twice* as an addend, we have $+10$; *three* times, $+15$; *four* times, $+20$; *five* times, $+25$. Thus,

$$3 \cdot (+5) = +15, \qquad 4 \cdot (+5) = +20, \qquad 6 \cdot (+8) = +48,$$

which are the same as

$$(+3)(+5) = +15, \qquad (+4)(+5) = +20, \qquad (+6)(+8) = +48.$$

Taking -5 *twice* as an addend, we have -10; *three* times, -15; *four* times, -20; *five* times, -25. Thus,

$$4 \cdot (-5) = -20, \qquad 7 \cdot (-5) = -35, \qquad 9 \cdot (-5) = -45,$$

which are the same as

$$(+4)(-5) = -20, \qquad (+7)(-5) = -35, \qquad (+9)(-5) = -45.$$

A **negative multiplier** means that the product is of the opposite quality from what it would be if the multiplier were positive. Therefore,

$$(+5)(-4) = -20 \qquad (-7)(-6) = +42 \qquad (-8)(-5) = +40$$

From the foregoing examples,

$$(+6)(+5) = +30 \qquad (+7)(-5) = -35$$
$$(-6)(-6) = +36 \qquad (-8)(+6) = -48$$

From these results we may derive a law of signs for multiplying *positive* and *negative* numbers.

127. Sign Law of Multiplication.— *Like signs of two numbers give a positive product, and unlike signs give a negative product.*

128. *The product of two or more numbers must contain as factors all the factors of each of the numbers.*

Thus, $$2a \times 3b = 2 \cdot 3 \cdot a \cdot b = 6ab$$

Exercise 43

Give the products of the following orally:

1. $\begin{array}{r}3x\\ \underline{2y}\end{array}$	**2.** $\begin{array}{r}-ab\\ \underline{3c}\end{array}$	**3.** $\begin{array}{r}4a\\ \underline{-3n}\end{array}$	**4.** $\begin{array}{r}-6x\\ \underline{yz}\end{array}$	**5.** $\begin{array}{r}-5a\\ \underline{-3x}\end{array}$
6. $\begin{array}{r}5a\\ \underline{4b}\end{array}$	**7.** $\begin{array}{r}-xy\\ \underline{2z}\end{array}$	**8.** $\begin{array}{r}-6a\\ \underline{-bc}\end{array}$	**9.** $\begin{array}{r}4b\\ \underline{-3c}\end{array}$	**10.** $\begin{array}{r}-7b\\ \underline{-3a}\end{array}$

When a term contains a twice as a factor, it is not written aa, but a^2, and is read: *a square.*

When a term contains x 3 times as a factor, it is not written xxx, but x^3, and is read: *x cube.*

129. An **exponent** is a symbol of number written at the right and a little above another symbol of number to show how many times the latter is *taken as a factor.*

$$2ab^2c^3 = 2 \cdot a \cdot b \cdot b \cdot c \cdot c \cdot c = 2abbccc$$

This is its signification only when the exponent is a positive integer.

It must be remembered that when no exponent is expressed the exponent 1 is always understood. Thus abx means $a^1b^1x^1$.

Observe that $a^5 = a \times a \times a \times a \times a,$

while $5a = a + a + a + a + a$

Exercise 44

Give the following products:

1. $6a^2x$ $3x^2y$	2. $3a^3x$ $-5a^2x$	3. $6xy^2$ $-ax^2$	4. $-xy^2$ $7y^2z$	5. $-7a^3b$ $-2b^2c$
6. $4a^2b$ $3ab^3$	7. $-4ax^3$ $4a^3x$	8. $-3b^2c$ $-4a^2b$	9. $8ax^2$ $-4x^2y$	10. $-9x^2y$ $-2x^3y$
11. $8ax^2$ $5a^3b$	12. $-7ax^2$ $2bx^2$	13. $7ab^3$ $-2a^3b$	14 $5a^2x$ $-9x^3y$	15. $-8a^2x$ $-2b^3x$

There are three important fundamental laws of multiplication which it will be well to notice here.

These are: *law of order*, or *commutative law; law of grouping*, or *associative law;* and *distributive law.*

134. Law of Order.— *The product of several numbers is the same in whatever order they are used.*

It is evident that

$$8 \cdot 5 \cdot 3 = 5 \cdot 3 \cdot 8 = 3 \cdot 8 \cdot 5$$

for each member of this equality is the same number.

In general numbers,

$$\mathbf{a \cdot b \cdot c = b \cdot c \cdot a = c \cdot a \cdot b}$$

135. Law of Grouping.— *The product of several numbers is the same in whatever manner they are grouped.*

$8 \cdot 5 \cdot 3$ denotes that 8 is to be multiplied by 5 and the product multiplied by 3; that is, $8 \cdot 5 \cdot 3 = (8 \cdot 5) \cdot 3$.

By the law of order,

$$8 \cdot 5 \cdot 3 = 5 \cdot 3 \cdot 8 = 3 \cdot 8 \cdot 5$$

Therefore,

$$8 \cdot 5 \cdot 3 = (8 \cdot 5) \cdot 3 = (5 \cdot 3) \cdot 8 = (3 \cdot 8) \cdot 5$$

In general numbers,

$$\mathbf{a \cdot b \cdot c = (a \cdot b) \cdot c = (b \cdot c) \cdot a = (a \cdot c) \cdot b}$$

136. Distributive Law.— *The product of a polynomial and a monomial is the algebraic sum of the products obtained by multiplying each term of the polynomial by the monomial.*

$$(8+7)\cdot 6=8\cdot 6+7\cdot 6$$

In general numbers,

$$(\mathbf{b}+\mathbf{c})\cdot \mathbf{a}=\mathbf{ab}+\mathbf{ac}$$

This is called the distributive law, because the multiplier is distributed over the terms of the multiplicand.

137. A **power** is the product obtained by taking a number any number of times *as a factor.*

138. A **square,** or **second power,** is the product obtained by taking a number *twice* as a factor. Thus,

$5^2=5\cdot 5=25$ $\quad 7^2=7\cdot 7=49$ $\quad (6a)^2=6a\cdot 6a=36a^2$

139. A **cube,** or **third power,** is the product obtained by taking a number *three* times as a factor.

$5^3=5\cdot 5\cdot 5=125$ $\quad (3a^2)^3=3a^2\cdot 3a^2\cdot 3a^2=27a^6$

The repeated factor is the *root* of the power, and the exponent indicating the power is the *exponent of the power.* The product is the *power.* Thus,

$$\text{root}\rightarrow 2^{\overset{\text{exponent}}{\downarrow}3}=8\leftarrow\text{power}$$

POWERS OF MONOMIALS

140. To find a power of any number is simply to find the product of two or more equal factors. Thus,

$$(2ab^2)^4=2ab^2\cdot 2ab^2\cdot 2ab^2\cdot 2ab^2=16a^4b^8$$

By the law of signs in multiplication, § 127, *all powers of positive numbers and* ***even*** *powers of negative numbers are* ***positive; odd*** *powers of negative numbers are* ***negative.***

141. Rule.— (1) *Raise the numerical coefficient to the required power,* (2) *multiply the exponent of each letter by the exponent of the power, and* (3) *give the result the proper sign.*

Exercise 45

Give these indicated powers:

1. $(2a)^2$	**2.** $(-2c^2)^3$	**3.** $(-\frac{1}{2}x^3)^2$	**4.** $(a^3x^2)^3$
5. $(3x)^3$	**6.** $(-4a^3)^2$	**7.** $(-\frac{1}{3}a^2)^3$	**8.** $(a^2x^3)^2$
9. $(2y)^3$	**10.** $(-3x^2)^4$	**11.** $(-\frac{1}{4}a^3)^2$	**12.** $(a^4x^4)^2$
13. $(2a)^4$	**14.** $(-2a^2)^5$	**15.** $(-\frac{1}{2}a^2)^3$	**16.** $(x^3y^3)^3$
17. $(7x)^2$	**18.** $(-5x^2)^2$	**19.** $(-\frac{1}{3}x^3)^3$	**20.** $(a^2b^2)^4$
21. $(3a)^3$	**22.** $(-4a^2)^4$	**23.** $(-\frac{2}{3}a^2)^2$	**24.** $(x^5y^5)^2$

MULTIPLYING A POLYNOMIAL BY A MONOMIAL

142. Observe carefully:

$$\begin{array}{l} 3a^4b^3-2a^2b^3+3a^2b-2ab^2 \\ \underline{2ab^2 \qquad\qquad\qquad\qquad\qquad} \\ 6a^5b^5-4a^3b^5+6a^3b^3-4a^2b^4 \end{array}$$

143. Rule.—*Multiply each term of the multiplicand by the multiplier as in multiplication of monomials.*

Exercise 46

Multiply:

1. $3ax^2+4a^3x$ by $3a^2x^3$	**2.** $3a^2b^3-ab^2+3a^2b^2$ by $4a^3b^2$
3. $5x^2y-3xy^2$ by $4x^3y^2$	**4.** $5a^3n^2-a^2n-4a^3n^3$ by $5a^2n^2$
5. $3ac^3-4a^2c$ by $5a^3c^3$	**6.** $6a^2b^3-ab^3+3a^3b^2$ by $6a^2b^3$
7. $6a^3x-7ax^2$ by $3a^3x^2$	**8.** $5b^3c^2+b^2c-4b^2c^2$ by $3b^2c^2$

Exercise 48

Multiply:

1. $12x^2-4+2x$ by $4+5x^2-3x$
2. $3a-2+4a^2$ by $4a+3a^3-2a^2$
3. $2x+3+x^2$ by $3x^2-2x+3-x^3$
4. $2ab-3b^2+a^2$ by $2a^2+3ab-b^2$
5. $3a^2-2b^2+3c^2$ by $4a^2-2b^2+4c^2$
6. $4ac-3a^2+2c^2$ by $2a^2-c^2+3ac$
7. $x^3+2x^2y+4xy^2$ by $2y+y^2-4x^2$
8. $3a^3-3a+2a^2-4$ by $5-3a^2-3a$
9. $2xy^2-x^3+3x^2y-y^3$ by $y^2-3xy+x^2$
10. $3a^4+2a-3a^2-a^3+3$ by $3a-a^2+4$
11. $4x^3-3x-4x^2+1$ by $3x^3+x-3x^2-5$

Perform the following indicated multiplications and unite results into as few terms as possible:*

12. $(a+c+x)2(a+c-x)-(2ac-x^2)^2$
13. $3(a-2c)(a+2c)2-5(a-3c)^2+69c^2$
14. $2(3a+x)^2-(3a-3x)(3a+3x)-11x^2$
15. $(a-2c)(c-3a)-(3a+c)(2c-a)-2ac$
16. $(2x+3y)2(2x-3y)+2(2x+3y)^2-19xy$
17. $7(x+3y)(x-3y)-(x-5y)^2-6(x^2-4y^2)$
18. $4(x+3)(x+2)+(x-6)(x+4)-3x(x+7)$

*First decide how many terms there are in each of the given exercises.

Exercise 49 — Special Products

Perform the following indicated operations as rapidly as you can, using pencil only when necessary:

1. $(a+1)(a+1)=$
2. $(a+5)(a+5)=$
3. $(a+x)^2=$
4. $(2x+1)^2=$
5. $(7+x)^2=$
6. $(x+4)(x+3)=$
7. $(x+12)(x+3)=$
8. $(8+a)(5+a)=$
9. $(x+a)(x+b)=$
10. $(m+r)(m+s)=$
11. $(a^2+b^2)(a^2-b^2)=$
12. $(x^3+y^3)(x^3-y^3)=$
13. $(x^4+y^4)^2=$
14. $(x+y)(x^2-xy+y^2)=$
15. $(2x+1)^2=$
16. $(3x+2y)^2=$
17. $(a-b)(a^2+ab+b^2)=$
18. $(a-b)(a^2-2ab+b^2)=$
19. $(x+y)(x^2+xy+y^2)=$
20. $(x-1)(x+1)(x^2+1)=$
21. $(a+b)^3=$
22. $(a+b+x)(a+b-x)=$
23. $(a-1)(a-1)=$
24. $(a-5)(a-5)=$
25. $(a-x)^2=$
26. $(2x-1)^2=$
27. $(7-x)^2=$
28. $(x+4)(x-3)=$
29. $(x-12)(x-3)=$
30. $(8-a)(5-a)=$
31. $(x-a)(x-b)=$
32. $(m+r)(m-s)=$
33. $(a^2+b^2)^2=$
34. $(x^3+y^3)^2=$
35. $(x^4-y^4)^2=$
36. $(x-y)(x^2+xy+y^2)=$
37. $(2x-1)^2=$

CHAPTER X

SIMPLE EQUATIONS

147. The **degree of a term** is indicated by the *sum* of the exponents of the literal factors.

Thus, a^2x^2 is a term of the *fourth degree.*

The degree of a term *in any particular letter* is indicated by the exponent of that letter in the term.

Thus, a^2x^2 is of the *second degree in* x.

148. The **degree of an equation** in one unknown is the degree of the *highest power* of the unknown number.

$5x+7=2x-a$ is an equation of the *first degree.*

$x-b=4x^2-3$ is an equation of the *second degree.*

149. A **simple equation,** or **linear equation,** is an equation which, when cleared and simplified, is of the *first* degree.

Whether or not a fractional equation is a simple equation cannot be determined until it is cleared of fractions and the resulting equation reduced to its simplest form.

Also, $x^2+x-4=x^2+3$ and $2x^2+x+5=x^2+x(x+2)$ are simple, or linear equations.

These are simple equations, because when similar terms are united, the square of the unknown number disappears.

150. **Checking** or **verifying** a root of an equation is the process of proving that the root satisfies the equation.

This is done by substituting the root found in the equation and ascertaining whether the result is an *identity.*

8. $5(x+3)-4(x-2)-3(2+x)-7=0$

9. $5x-3(x-6)=2(8-x)-4(9+x)+6$

10. $\frac{x}{4}+x-3\frac{3}{8}=\frac{5x+5}{5}$

11. $\frac{11x+9}{8}-x-1\frac{2}{3}=\frac{x}{4}$

12. $4\frac{1}{3}-x+\frac{11x-5}{6}=\frac{4x}{9}$

13. $\frac{x}{4}-x-3\frac{1}{5}=\frac{11x+5}{10}$

In checking or verifying the solution of a problem, the substitution should be made in the *problem itself*.

14. A has twice as much money as B, and B has twice as much as C. If all have \$595, how much has C?

15. The sum of the third, fourth, and eighth parts of a number is 68. Find the number.

16. The length of a rectangle is twice its width, and the perimeter is 144 feet. Find the dimensions.

17. A man gave \$125 to his 5 sons, each of 4 of them receiving \$5 more than his next younger brother. How much did the oldest son receive?

18. James has $\frac{1}{3}$ as many marbles as Frank. If James buys 120 and Frank loses 23, James will then have 7 more than Frank. How many has each?

19. The sum of two numbers is 85, and 3 times the smaller exceeds twice the larger by 20. Find the larger number.

20. Seven men agreed to share equally in buying a boat, but, as 3 of them were unable to pay, each of the others had to pay \$30 more than his original share. Find the cost of the boat.

21. Three men invested \$9400 in business. A put in \$600 more than B, and C invested \$200 less than A. How much did A and C together invest in the business?

22. A farmer sold 30 lambs and 60 sheep for \$300. He received twice as much per head for the sheep as for the lambs. How much did he receive for the 60 sheep?

152. *In clearing an equation of fractions, if a fraction is preceded by the minus sign, the sign of each term of the numerator must be changed, for the fraction-line is a vinculum for the numerator.*

Thus, $$\frac{x}{2}-\frac{x+12}{3}=\frac{2-x}{8}.$$

gives, $$12x-8x-96=6-3x$$

Exercise 51 — Equations and Problems in One Unknown

Solve the following, checking some of them:

1. $\frac{3+2x}{9}+\frac{x-8}{3}+2\frac{1}{3}=\frac{4x+5}{9}-\frac{x-5}{6}$

2. $\frac{2x+2}{4}-\frac{2x-3}{2}+7\frac{1}{2}=\frac{x}{3}-x+\frac{4x-3}{2}$

3. $\frac{x-8}{4}+2x-\frac{5x-12}{4}=\frac{2x-6}{2}+\frac{2x-4}{5}$

4. $\frac{5x+15}{2}-\frac{2x-5}{3}-7\frac{2}{3}=\frac{3x+8}{3}-\frac{2x-7}{6}$

5. $\frac{x+7}{2}-\frac{5(x-2)}{4}+2(x-3)=6\frac{3}{4}+\frac{3(x-6)}{2}$

6. $\frac{3x+8}{2}-9\frac{2}{3}-\frac{2(x-3)}{3}=\frac{4(x-2)}{3}-\frac{3(x+2)}{2}$

7. $\frac{3x+4}{2}-7\frac{1}{2}+2(x+3)=\frac{3(x+6)}{4}+\frac{3(4x+6)}{6}$

8. Nine boys and 16 men earn \$365 a week. If each man earns 4 times as much as each boy, how much do the 9 boys earn per week?

9. A boy has \$3.60 in dimes and 5-cent pieces, and he has 4 times as many 5-cent pieces as dimes. How many coins has he and what is the value of each kind?

10. A walked 95 miles in 3 days, going 4 miles more the second day than the first and 3 miles more the third day than the second. How far did he go the third day?

11. A is 3 times as old as B. Ten years ago A was 5 times as old as B. Find A's age now.

Let x = the number of years in B's age now.
and $3x$ = the number of years in A's age now.
$x-10$ = the number of years in B's age 10 years ago,
$3x-10$ = the number of years in A's age 10 years ago.

$$5(x-10)=3x-10$$

12. A is 4 times as old as his son, and 5 years ago he was 7 times as old. Find the father's age.

13. A man is 24 years older than his son. Fourteen years ago he was 3 times as old. Find the age of each.

14. A farmer sold corn, wheat, and oats. For his corn and wheat he received \$800. For his corn and oats he received \$720, and for his wheat and oats \$840. How much did he receive for all his grain?

15. A man spent $\frac{1}{8}$ of his money for a suit of clothes, $\frac{1}{4}$ of it for a watch, and had \$115 left. How much did he spend?

16. The sum of two numbers is 82, and if the greater is divided by the less, the quotient is 5 and the remainder 4. Find the two numbers.

17. D is 6 years older than C; C is 4 years older than B; B is 3 years older than A. If they live 5 years, the sum of their ages will be 135 years. Find D's age.

18. A grocer mixed tea worth 70¢ a pound with tea worth 50¢ a pound in such proportions that the mixture weighing 100 pounds was worth \$58. How many pounds of each kind were in the mixture?

Exercise 52 — Problems in Simultaneous Equations

Solve the following problems in simultaneous simple equations:

1. The sum of two numbers is 85, and their difference exceeds $\frac{1}{5}$ of the smaller by 8. Find the numbers.

Let x = the larger number,
and y = the smaller number.

$$x+y=85$$

$$x-y-8=\frac{y}{5}$$

The second equation contains a fraction. Clear this of fractions and then with the other equation, eliminate.

2. If 5 is added to the numerator of a certain fraction, its value is $\frac{4}{5}$; and if 1 is subtracted from the denominator, its value is $\frac{1}{3}$. Find the fraction.

Let n = the numerator,
and d = the denominator.

3. Three times the larger of two numbers exceeds $\frac{1}{3}$ of the smaller by 66, and 3 times the smaller exceeds $\frac{1}{3}$ of the larger by 46. Find the numbers.

4. If 3 is added to both terms of a certain fraction, its value is $\frac{2}{3}$; and if 4 is subtracted from both terms, its value is $\frac{3}{8}$. Find the fraction.

5. A miller bought 50 bushels of corn and 40 bushels of oats for $64. At another time he bought at the same prices 38 bushels of oats and 70 bushels of corn for $78.80. How much did he pay for all of the corn?

6. A dealer bought oranges, some at 2 for 5¢ and some at 3 for 5¢, paying $12 for all. Three dozen were unsalable, but he sold the remainder at 30¢ a dozen, making a profit of $2.10. How many oranges did he buy?

CHAPTER XI

DIVISION

153. **Division** is the process of finding *one* of two numbers when their *product* and the *other* number are known.

154. The **dividend** is the number to be divided and represents the product of the two numbers.

155. The **divisor** is the number by which we divide and represents *one factor* of the dividend.

156. The **quotient** is the number obtained by division and represents the *other factor* of the dividend.

Since division is the reverse of multiplication, the rule for division is derived from the process of multiplication.

Three things must be determined: The *sign of the quotient*, the *coefficient*, the *exponent of each letter.*

DIVIDING A MONOMIAL BY A MONOMIAL

157. The Sign of the Quotient.

$$(+7)(+5)=+35, \text{ therefore } (+35)\div(+5)=+7$$
$$(+7)(-5)=-35, \text{ therefore } (-35)\div(-5)=+7$$
$$(-7)(-5)=+35, \text{ therefore } (+35)\div(-5)=-7$$
$$(-7)(+5)=-35, \text{ therefore } (-35)\div(+5)=-7$$

158. Sign Law of Division.—*Like signs of dividend and divisor give a positive quotient; unlike signs, a negative quotient.*

Give the following quotients:

$(+56)\div(+7)$ $(-64)\div(-8)$ $(96)\div(-8)$ $(-63)\div(+9)$
$(+84)\div(-7)$ $(-72)\div(-9)$ $(68)\div(-4)$ $(+75)\div(-5)$

Since $5a\times 3x=15ax$, therefore $15ax\div 3x=5a$

The coefficient of the quotient is the coefficient of the dividend divided by the coefficient of the divisor.

159. The Exponent in the Quotient. Since the dividend is a product, one factor of which is the divisor, the exponent of the dividend is the sum of the exponents of divisor and quotient.

To find the exponent of the quotient, *subtract* the exponent of the *divisor* from that of the *dividend.*

160. Law of Exponents for Division.—*Each exponent in the divisor is subtracted from the exponent of the same letter in the dividend.*

Since $a^3\times a^2=a^5$, therefore $a^5\div a^2=a^3$

In general numbers,

$$\mathbf{a}^m\div\mathbf{a}^n=\mathbf{a}^{m-n}$$

Observe the following:

$$2ab\overline{)8a^3b^2}\qquad -2a^2b\overline{)-8a^4b^3}\qquad -3a^2b\overline{)12a^3b^2c}$$
$$4a^2b\qquad 4a^2b^2\qquad -4abc$$

By the law of exponents for division, $a^3\div a^3=a^0$. But any number, except 0, divided by itself *also equals* 1. Therefore,

$$\mathbf{a^0=1.}$$

161. Meaning of Exponent 0. Since a may represent any number, it follows that any number with a *zero-exponent* is equal to 1. Thus,

$$2abc^0=2ab\cdot 1=2ab$$

From this equation it is evident that any letter with a zero-exponent may be omitted from a term, because its presence only multiplies the rest of the term by 1.

DIVIDING A POLYNOMIAL BY A POLYNOMIAL

163. The rule for dividing a polynomial by a polynomial is deduced from the process of multiplication.

Study this example carefully:

$$\begin{array}{l|l} a^4+2a^3b-6a^2b^2+26ab^3-15b^4 & a^2+4ab-3b^2 \\ \cline{2-2} \underline{a^4+4a^3b-3a^2b^2} & a^2-2ab+5b^2 \\ \quad -2a^3b-3a^2b^2+26ab^3 & \\ \quad \underline{-2a^3b-8a^2b^2+\ \ 6ab^3} & \\ \qquad\qquad 5a^2b^2+20ab^3-15b^4 & \\ \qquad\qquad \underline{5a^2b^2+20ab^3-15b^4} & \end{array}$$

Arrange the dividend and divisor with reference to the descending powers of a, writing the divisor at the right of the dividend.

Since the dividend is the product of the divisor and quotient, it is the algebraic sum of the products obtained by multiplying the divisor by the several terms of the quotient.

Hence, when dividend, divisor, and quotient are arranged with reference to the descending powers of some letter, the first term of the dividend is the product of the first terms of the divisor and quotient, whence the first term of the quotient is the quotient of the first term of the dividend divided by the first term of the divisor.

Dividing the first term of the dividend by the first term of the divisor, we have a^2 for the first term of the quotient.

Since the dividend is the algebraic sum of the products obtained by multiplying the divisor by the several terms of the quotient, if the product of the divisor and first term of the quotient is subtracted from the dividend, the remainder, which is a new dividend, is the product of the divisor and the other terms of the quotient, and the next term of the quotient is the quotient of the first term of the remainder divided by the first term of the divisor.

Dividing the first term of the remainder by the first term of the divisor, we have $-2ab$ for the second term of the quotient.

Repeating this process until there is no remainder, we obtain the quotient $a^2-2ab+5b^2$.

Each remainder must be arranged in the same manner as the dividend and divisor.

Observe the following solutions:

(I)

$$
\begin{array}{l|l}
a^4-a^2b^2+2ab^3-\quad b^4 & a^2-ab+b^2 \\ \cline{2-2}
\underline{a^4-a^3b+\ a^2b^2} & a^2+ab-b^2 \\
\quad a^3b-2a^2b^2+2ab^3 & \\
\quad \underline{a^3b-\ a^2b^2+\ ab^3} & \\
\qquad -\ a^2b^2+\ ab^3-b^4 & \\
\qquad \underline{-\ a^2b^2+\ ab^3-b^4} &
\end{array}
$$

(II)

$$
\begin{array}{l|l}
x^4+4y^4 & x^2-2xy+2y^2 \\ \cline{2-2}
\underline{x^4-2x^3y+2x^2y^2} & x^2+2xy+2y^2 \\
\quad 2x^3y-2x^2y^2+4y^4 & \\
\quad \underline{2x^3y-4x^2y^2+4xy^3} & \\
\qquad 2x^2y^2-4xy^3+4y^4 & \\
\qquad \underline{2x^2y^2-4xy^3+4y^4} &
\end{array}
$$

Exercise 55 — Dividing Polynomials

Divide:

1. a^2-a-42 by $a+6$
2. x^2-x-30 by $x-6$
3. a^2+a-72 by $a+9$
4. x^2+x-56 by $x-7$
5. $a^2-6a-16$ by $a+2$
6. $x^2+8x-33$ by $a-3$
7. $a^2-9a-52$ by $a+4$
8. $x^2+7x-98$ by $x-7$
9. $a^2+32a+60$ by $a+2$
10. $x^2-17x-18$ by $1+x$
11. $a^2+25a-54$ by $a-2$

12. $x^2+20x+75$ by $x+15$

13. a^4-15a^2+56 by a^2-7

14. $3a^6-8a^3-28$ by a^3+2

15. $5x^8+42x^4+85$ by x^4+5

16. $a^4-ac^3-3a^3c+c^4$ by $a-c$

17. $b^3+b^2x-bx^2-x^3$ by $b-x$

18. $a^3+ax^2+a^2x+x^3$ by $a+x$

19. $a^4+b^4+a^2b^2$ by a^2+b^2-ab

20. $ax^3+abx+bx^2+b^2$ by $ax+b$

21. $x^3+3xy^2+3x^2y+y^3$ by $x+y$

22. $x^2-y^2+z^2-2xz$ by $x-y-z$

23. $a^2+2xy-y^2-x^2$ by $a+x-y$

24. $32x^3-6x-1$ by $8x^2-2x-1$

25. $4a^4+6a^2+8a^3-24$ by $2a+4$

26. a^3+27 by $a+3$

27. $36x^2-81y^2$ by $6x+9y$

28. a^4-16 by $a-2$

29. $27x^3+64y^3$ by $3x+4y$

30. a^3-64 by $a-4$

31. $25x^2-16y^2$ by $5x-4y$

32. a^4-81 by $a+3$

33. $8x^3-125y^3$ by $2x-5y$

34. a^6+27 by a^2+3

35. $16x^2-64y^2$ by $4x+8y$

36. a^9-64 by a^3-4

37. $16x^4-81y^4$ by $2x+3y$

38. a^8-16 by a^2+2

39. $25x^2-49y^2$ by $5x-7y$

can do that; but express in symbols all you can of the conditions of the problem, no matter how useless this may seem. From these expressions you will see what numbers are equal, and the formation of the equation will become a simple matter.

VII. It is much easier to reason about small numbers than about large ones. If the numbers in a problem are large, or complicated, or are general numbers, simplify the problem by replacing them with simple arithmetical numbers; then reread the problem using the simple numbers, and try again to sense the meaning. To form the habit of doing this will help you greatly.

VIII. School work that requires little or no effort on your part will not increase your power to do harder things. You should welcome some tasks that test you to the limit; and if you would grow stronger, *you must always rely upon yourself.* It is ruinous to your progress to rely on others to assist you in solving your problems.

IX. Appeal to your teacher for assistance only after you have really done your best, and then ask only for one or two hints to start you right.

Exercise 56 — Problems Requiring Simple Equations

Solve the following equations and problems:

1. $\frac{x+5}{3}-\frac{x+2}{6}+\frac{x-3}{7}=4$

2. $\frac{s+3}{4}+\frac{s+1}{3}-\frac{s+8}{5}=6$

3. $\frac{7n+4}{5}+3n-\frac{3n-9}{3}=31$

4. $4x-\frac{5x-8}{7}-17=\frac{8x+5}{5}$

5. $\frac{2y-5}{6}+\frac{35}{2}=5y-\frac{6y+3}{4}$

6. $\frac{n+9}{2}-\frac{n+5}{4}=n-\frac{n+9}{3}$

7. $\frac{s+5}{6}+\frac{s-1}{9}=s-\frac{s+7}{2}$.

8. The sum of two numbers is 94, and their difference is 38. Find the numbers.

9. A boy has 3 times as many dimes as quarters, and he has \$11 in all. How many coins has he?

10. Seven times a certain number is 176 more than 3 times the number. Find the number.

11. A man bought 50 sheep, some at \$3.75 a head and the others at \$4.50 a head. The average cost was \$4.05. How many did he buy at the lower price?

12. A boy earns \$1.25 a day less than his father, and in 14 days the father earns \$15 more than the son earns in 16 days. How much do both earn per day?

13. A clerk spends $\frac{1}{4}$ of his annual salary for board, $\frac{1}{8}$ for clothes, $\frac{1}{6}$ for other expenses, and saves \$1100. How much are his annual expenses?

14. At what rate per annum will \$8000 yield \$540 interest in 1 year and 6 months?

Let $x =$ the rate per annum.

$$8000 \times \frac{x}{100} \times \frac{3}{2} = 540$$

15. A man invested a certain sum at 5% and twice as much at 6%. His annual income from both investments was \$680. How much did he invest?

16. A is 64 years old, and B is $\frac{3}{4}$ as old. How many years have passed since B was $\frac{1}{2}$ as old as A?

17. The sum of two numbers is 84, and 7 times the less exceeds 5 times the greater by 12. Find the numbers.

18. A had 8 acres of land less than B, but A sold 24 acres to B. A then had left only $\frac{1}{2}$ as many acres as B. How many acres did each have at first?

19. A woman bought 36 yards of silk for \$31, paying 75¢ a yard for part of it and \$1 a yard for the rest. How many yards of each kind did she buy?

20. A grocer has tea worth 40¢ a pound and some worth 60¢ a pound. How many pounds of each must he take to mix 60 pounds worth 54¢ a pound?

Solve the 20th with one and then with two unknown numbers.

21. A boy bought a number of apples at the rate of 7 for 10¢ and sold them at the rate of 10¢ for 3, gaining \$2. How many apples did he buy?

22. If it costs the same at \$1 a yard to enclose a square court with a fence as to pave it at 10¢ a square yard, what are the dimensions of the court?

23. A mason received \$3.60 a day for his labor and paid 85¢ a day for his board. At the end of 44 days he had saved \$92.20. How many days did he work?

24. A, B, and C together earn \$5000. A's salary is $\frac{2}{3}$ of B's and \$450 less than C's. Find C's salary.

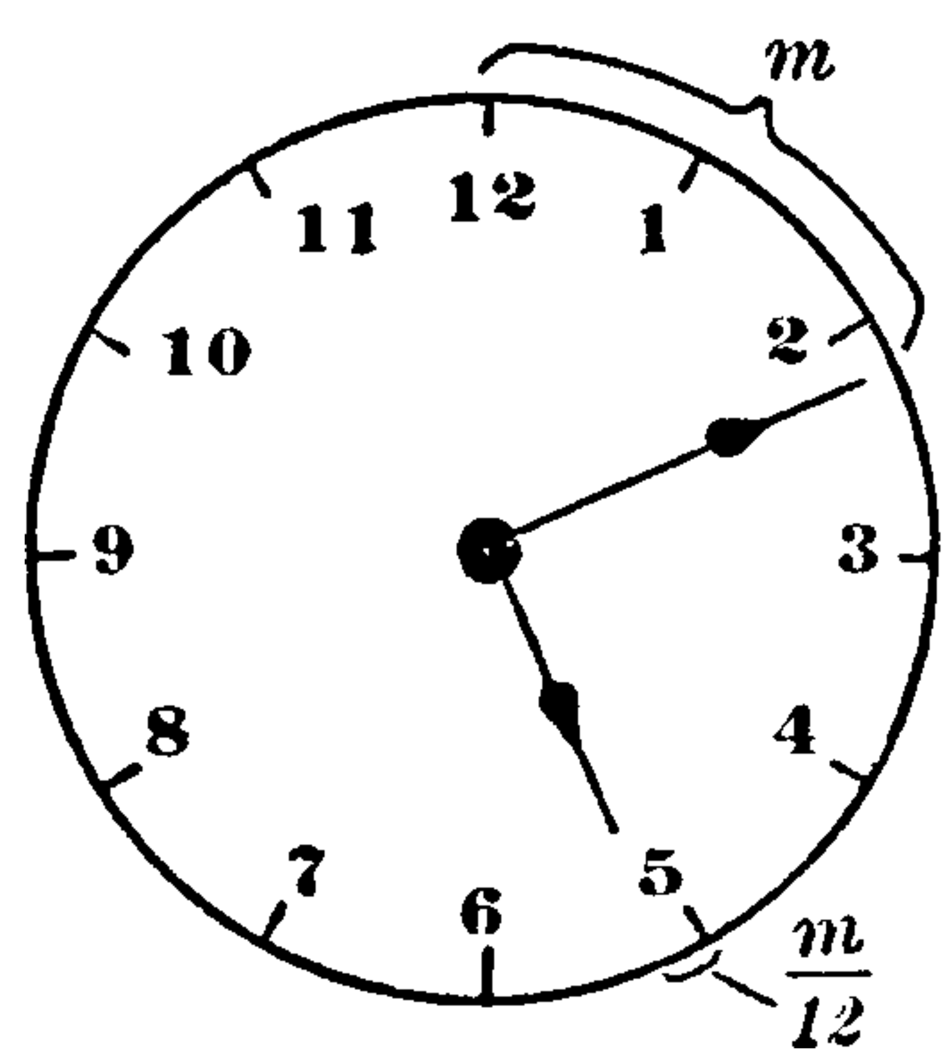
25. The sum of $\frac{1}{4}$ and $\frac{1}{5}$ of a number exceeds 5 times the difference between $\frac{1}{6}$ and $\frac{1}{8}$ of the number by 29. Find the number.

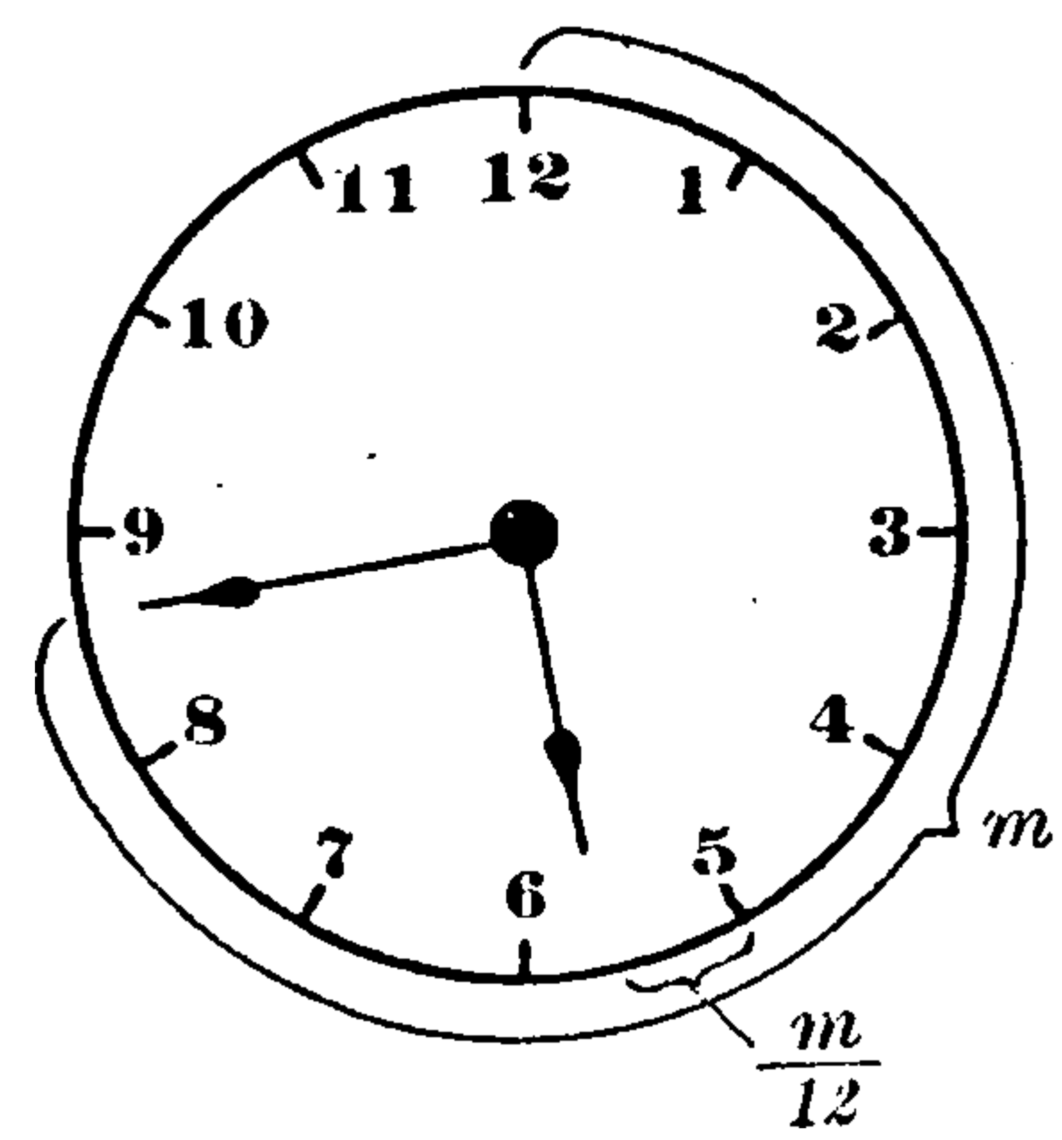
26. If $\frac{2}{5}$ of a certain principal is invested at 4% and the remainder at 5%, the annual income is \$690. Find the whole sum invested.

27. A bought sheep at \$4 a head and had \$33 left. If he had bought them at \$4.75 a head, he would have needed 75¢ more to pay for them. How many did he buy?

28. The length of a rectangle exceeds its width by 13 inches. If the length were diminished 7 inches and the width increased 5 inches, the area would remain the same. What are the dimensions of the rectangle?

34. At what times between 5 and 6 o'clock are the hands of a clock at right angles to each other?





The hands are at right angles twice between 5 and 6, once before the minute-hand passes the hour-hand, and once after.

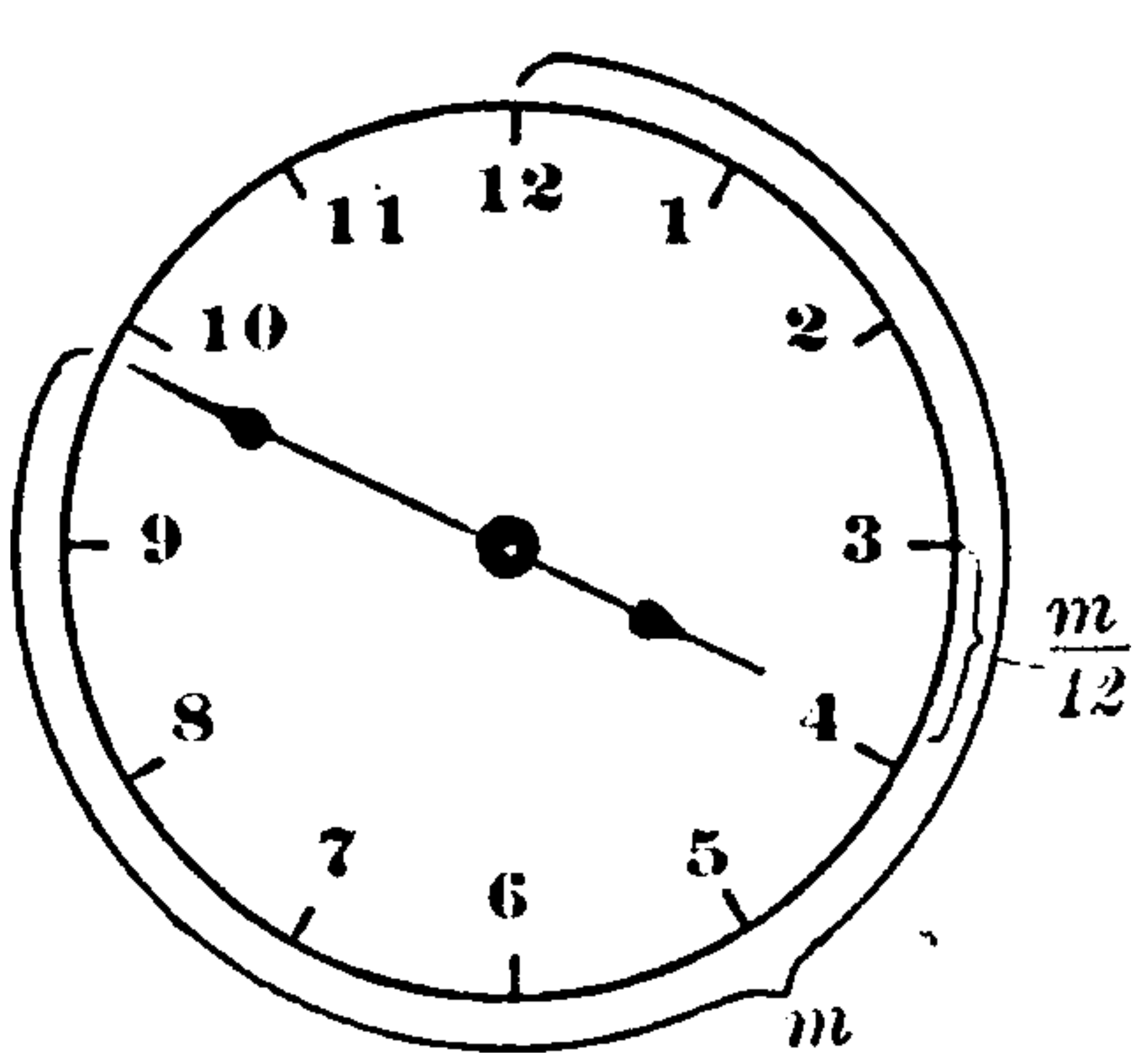
In the first case, the minute-hand must pass over 25 spaces, plus the number of spaces passed over by the hour-hand, *minus* 15 spaces.

In the second case, the minute-hand must pass over 25 spaces, plus m divided by 12, *plus* 15 spaces. The two equations are

$$m=25+\frac{m}{12}-15 \qquad\qquad m=25+\frac{m}{12}+15$$

35. At what time between 3 and 4 o'clock are the hands of a clock opposite each other?

$$m=15+\frac{m}{12}+30$$



36. At what time between 8 and 9 o'clock are the hands of a clock together?

37. At what time between 2 and 3 o'clock are the hands of a clock at right angles to each other?

38. A is 54 years old, and B is $\frac{1}{3}$ as old. In how many years will B be $\frac{1}{2}$ as old as A?

39. A, B, and C together earn \$3650. A's salary is $\frac{1}{2}$ of B's and \$650 less than C's. Find C's salary.

40. A boy has \$11 in half-dollars and 5-cent pieces, in all 58 coins. How many has he of each kind?

41. Find the number whose double diminished by 23 is as much greater than 53 as 68 is greater than the number.

42. A is 28 years older than his son, but 5 years ago he was 3 times as old. Find the father's age.

43. A man bought some cows at $40 a head. If he had bought 2 less for the same money, each would have cost $10 more. How many did he buy?

44. A had twice as many sheep as B. Each sold half his flock to C, and A sold 30 to B, whereupon A and B had the same number. How many had each at first?

45. One of two numbers is 4 times the other. If 24 is subtracted from the greater, and the less is subtracted from 66, the remainders are equal. Find the numbers.

46. A woman bought 12 yards of silk, but if she had bought 8 yards more for the same money, it would have cost 60¢ a yard less. How much did it cost?

47. A father and two sons earn $222 a month, the two sons receiving the same wages. If the sons' wages were doubled, they would together receive only $6 less than their father. How much does the father earn per month?

48. A man bought land at $90 an acre and had $1000 left. At $105 an acre, he would have needed $200 more to pay for it. How many acres did he buy?

49. A fruit dealer bought some oranges at the rate of 3 for 5¢ and twice as many others at the rate of 2 for 5¢. He sold them all at 36¢ a dozen and made a profit of $5.60. How many oranges did he buy?

50. A pedestrian walked a certain distance at the rate of $1\frac{3}{4}$ miles an hour. He rested 2 hours at the end of his journey and returned at the rate of $2\frac{1}{3}$ miles an hour. If he was out 9 hours, how many miles did he walk?

ELIMINATION BY SUBSTITUTION

165. The following example illustrates the method of **elimination by substitution:**

$$3x+2y=65 \qquad (1)$$

$$4x-3y=30 \qquad (2)$$

Transposing $2y$ in (1), $$3x=65-2y \qquad (3)$$

Dividing (3) by 3, $$x=\frac{65-2y}{3} \qquad (4)$$

Substituting in (2), $$\frac{260-8y}{3}-3y=30 \qquad (5)$$

Solving (5), we have the value of y, and substituting this value in (1) or (2), we find the value of x.

166. Rule.—*Determine first which of the two unknown numbers it is more convenient to eliminate.*

From either equation, find the value of that unknown number in terms of the other. Substitute this value for the same unknown number in the other equation.

Exercise 57

Eliminate by substitution and solve:

1. $\begin{cases} 4x-6y=6 \\ 2x+3y=9 \end{cases}$

2. $\begin{cases} 5x+4y=-4 \\ 4x+3y=-2 \end{cases}$

3. $\begin{cases} 3x-3y=9 \\ 4x-5y=7 \end{cases}$

4. $\begin{cases} 4x-5y=-2 \\ 3x-4y=-3 \end{cases}$

5. $\begin{cases} 4x+2y=5 \\ 5x+3y=7 \end{cases}$

6. $\begin{cases} 2x+2y=-5 \\ 6x+9y=-6 \end{cases}$

7. $\begin{cases} \frac{5x}{2}-\frac{3y}{2}=18 \\ \frac{3x}{4}+\frac{5y}{8}=14 \end{cases}$

8. $\begin{cases} \frac{3x}{7}-\frac{5y}{3}=-3 \\ \frac{9x}{14}-3y=-6 \end{cases}$

9. In 4 years a sum of money at simple interest amounts to $768, and in 5 years at the same rate it amounts to $800. Find the sum invested and the rate.

10. A pound of tea and 5 pounds of coffee cost $2. At prices 20% higher, 3 pounds of tea and 11 pounds of coffee would cost $6. Find the price of each.

11. If 7 is added to the sum of the two digits of a certain number, the result is 5 times the tens' digit, and if 45 is added to the number itself, the digits are interchanged. Find the number.

12. If the sum of two numbers is divided by 5, the quotient is 21 and the remainder 4; and if the difference of the numbers is divided by 10, the quotient is 6 and the remainder 3. Find the numbers.

13. A man paid $14 for oranges, buying some of them at 12 for 25¢ and the rest at 14 for 25¢. He sold them all at 30¢ a dozen and made a profit of $4.30. How many did he buy of each kind?

14. If the larger of two numbers is divided by the smaller, the quotient is 6 and the remainder 8; but if 7 times the smaller is divided by the larger, the quotient is 1 and the remainder 9. Find the numbers.

15. If the numerator of a certain fraction is doubled and 3 added to the denominator, its value is $\frac{4}{5}$; if the denominator is doubled and 2 added to the numerator, its value is $\frac{3}{7}$. Find the fraction.

16. If a rectangular plot of land were 20 feet longer and 10 feet wider, the area would be increased 3000 square feet; but if the length were 10 feet more and the width 30 feet less, the area would be diminished 2400 square feet. How many square feet are there in the plot?

CHAPTER XIII

GENERAL NUMBERS. FORMULAS. TYPE-FORMS

GENERAL NUMBERS

167. Representing Numbers. By common usage, the Arabic numerals of arithmetic and the letters used in algebra are called *numbers.* It must be remembered, however, that all number symbols are used simply to *represent* numbers.

Since letters are used in algebra to represent *any* numbers, these letters are called *general numbers.*

168. A **general number** is a letter or other number symbol that may represent any number.

To be able to read algebraic expressions in concise English and to express mathematical statements in algebraic symbols is of great importance.

For example, $3ab$, $3ax$, or $3xy$ represents three times the product of *any* two numbers. Also, $2(a-b)$ or $2(x-y)$ may represent twice the difference of *any* two numbers.

Since a and b may represent any two unequal numbers, the equality—

$$(a+b)-(a-b)=2b$$

expresses the following principle:

The sum of any two unequal numbers exceeds their difference by twice the smaller number.

If a and b are any two numbers of which b is the smaller, what principle does this equality express

$$2(a+b)-2(a-b)=4b?$$

What principles do the following identities express

$$(a+b)+(a-b)=2a \qquad (a+1)^2-a^2=2a+1?$$

FORMULAS

169. **A formula** is an expression of a *general principle*, or *rule* in general number symbols and in the form of an equality.

The expression of a formula in words is a *principle*, and the expression of it as a direction is a *rule*.

The ability to express general principles as formulas, and to read formulas accurately as principles and rules is of the greatest value to students of algebra, physics, etc.

The truth of the following algebraic statement, called a formula, may be verified by performing the indicated operations:

$$(x+y)^2-(x-y)^2=4xy$$

Supposing that x and y are any two numbers, what principle does the formula express?

Exercise 59

1. Verify the truth of this formula:

$$(a+x)^2-(a+x)(a-x)=2x(a+x)$$

2. Having verified the truth of this algebraic statement, tell what general principle it expresses.

Since a formula expresses a *general* principle, it applies to all *particular* examples of that type.

3. By how much does $687+125$ exceed $687-125$? By how much does $2(759+45)$ exceed $2(759-45)$?

4. How much does the square of $50+3$ exceed the square of $50-3$? Give result without squaring.

5. Without squaring the binomial, give the difference between $(20+6)^2$ and $(20+6)(20-6)$.

6. By how much does $569+350$ exceed $569-350$? By how much does $3(476+150)$ exceed $3(476-150)$?

2. When a rectangle 18 feet wide contains 150 square yards, what is its length?

$$l = \frac{9 \times 150}{18}$$

3. A rectangle of land 64 rods long contains 18 acres. Find its width in rods.

$$w = \frac{160 \times 18}{64}$$

4. Express in general numbers two rules for finding the perimeter of any rectangle.

5. Using any general numbers, write three formulas for finding the area of any triangle.

6. Solve one of the three formulas of problem 5 and give the rule which each of the derived formulas expresses.

7. If a triangle whose altitude is 24 feet contains 52 square yards, how long is its base?

8. If x is the age of a boy now, make the problem of which this equation is the statement:

$$x + 3 = 3(x - 7)$$

9. Using any general numbers, write the formula for finding the volume of any rectangular prism.

10. Solve the formula of problem 9 and give the principle which each of the three derived formulas expresses.

11. Express in a formula the relation of dividend, divisor, quotient, and remainder, in division.

12. A has x acres of land and B $3x$ acres. Make the problem of which the statement is $3x - 20 = 2(x + 20)$.

13. Without squaring the binomial, give the difference between $(75+3)^2$ and $(75+3)(75-3)$.

14. Give a formula for finding one dimension of a rectangle when the perimeter and the other dimension are given.

15. If a rectangle 64 feet long has a perimeter of 226 feet, what is the width?

16. Represent the number of cubic yards in any box-shaped excavation when the dimensions are given in feet.

172. The *formula as a compact shorthand* of number laws is perhaps the most practical part of algebra. The following list of problems will give practice in formulating arithmetical, practical, and scientific laws.

Exercise 61 — Stating and Formulating Laws

1. Denoting the minuend, subtrahend, and difference by m, s, and d, respectively, show by a formula the relation of these numbers.

2. Add s to both sides of $m-s=d$ and state what the resulting formula means.

3. Show by a formula the relation of the product, p, multiplicand, M, and multiplier, m.

4. Divide both sides of $p=M\cdot m$ by m, and state the meaning of the resulting formula.

5. State as a formula: "The product of a fraction, $\frac{n}{d}$, by a whole number, a, is the product of the whole number by the numerator, divided by the denominator."

6. Show by a formula the principle for multiplying a fraction, $\frac{a}{b}$, by a fraction $\frac{c}{d}$, calling the product p.

7. State by a formula the relation of the percentage, p, the rate, r, and the base, b, and translate the formula into words.

8. Divide both sides of $p=br$, by r, and tell the meaning of the resulting formula.

9. State and give meaning of the formula for the interest, i, in terms of the principle, p, rate, r, and time, t (in years).

10. Divide both sides of $i=prt$ by rt, and tell what the resulting formula means.

11. Divide both sides of $i=prt$ by pt, and tell what the resulting formula means.

12. State as a formula the law for subtracting two fractions.

13. State as a formula the law for multiplying two fractions.

14. Show by a formula the law of area, A, of a square of side, s.

15. State by a formula the volume, V, of a cube whose edge is s.

16. State by a formula the value, f, of a decimal fraction having t units in tenths' place and h units in hundredths' place.

Ans. $f=\frac{t}{10}+\frac{h}{100}$.

17. Solve the formula in the answer of problem 16 for t; for h.

18. State as a formula the cost-law, in which c is the total cost, n the number of articles, and p the price of each. Solve the formula for n; for p.

19. Calling d the total distance, r the rate of movement, and t the time, state the distance-law for uniform motion, as a formula.

20. Solve the formula of problem 19 for r, and tell the meaning of the result. Solve for t.

21. The velocity, v, of a freely falling body is the product of the gravity-constant, g, by the time, t, of fall. Formulate this law. Solve it for g; for t.

22. Solve the formula, $A=2\pi r(h+r)$ for π; for πr; for $h+r$; for h.

FORMS AND TYPE-FORMS OF ALGEBRAIC NUMBERS

173. Meaning of Type-Forms. A very important thing to learn in algebra is the meaning and use of **forms** and **type-forms** of algebraic numbers. By the form of a number is meant how, from its written appearance, it looks as though it were made up out of simpler numbers. A bit of valuable advice, often given, but seldom appreciated by the beginner, is *always to look carefully into a problem or exercise before putting pencil to paper.* "Look before you leap" is a good motto for the young algebraist. Make it a habit. The habit is particularly valuable in factoring. The amount of useless labor it will save you will compensate many-fold for the effort. The way to start the practice is to learn what number-forms mean and how to use them. This is not an entirely new thing, for number-forms are used early in arithmetic.

For example, when you learned to tell, without dividing, whether 5 is a factor of a number, by noticing whether it ended in 0 or 5, you were using the *form* of the number to lighten your work.

Likewise, you have probably learned to use the *form* of a number to decide, without dividing, whether the number is divisible by 10, 100, 2, 4, 8, etc.

In algebra, an acquaintance with *number-forms* is much more useful than in arithmetic.

If we were asked to indicate the sum or the difference of two different numbers in some suggestive form, we might write:

$$(\quad)+(\quad) \quad \text{and} \quad (\quad)-(\quad),$$

the empty curves suggesting that *any* numbers whatsoever might be written within them. But while these forms show *sum* and *difference*, they do not suggest that the two numbers in question are to be different numbers. To obviate this objection we might suggest these forms:

$$(\quad)+[\quad] \text{ and } (\quad)-[\quad],$$

with the understanding that the curved and the square-cornered symbols are to suggest that different numbers are to be written inside the differently-shaped symbols.

If we had been ingenious enough to see what it took mathematicians hundreds of years to discover, that by simply calling one number x and the other y, and writing,

$$x+y \text{ and } x-y,$$

we have everything shown easily and fully, then our problem would have been solved. We merely remember that the different letters are in general to denote different numbers.

174. Examples of Type-Forms. Any other letters, as a and b, might as well have been used as x and y in the last section. But x and y are easily written, and serve just as well as any other letters, so algebraists fall into the habit of using them more than others.

We say then that $x+y$ and $x-y$ are respectively the *forms* for the sum and the difference of any two different numbers. Since $x+y$ may stand for (typify) the *sum of any two numbers*, it may be called a *type-form* for the *sum*. Similarly, $x-y$ is called the *type-form* for the difference of two numbers.

The *type-form* for the sum of two products is $ax+by$; for the difference of two products, $ax-by$.

The *type-form* for the sum of two products having one factor *common* to both products is $ax+ay$, and for the difference of such products, $ax-ay$.

The *type-form* for the sum of two squares is x^2+y^2, and for the difference of two squares, x^2-y^2. Observe that x^2+y^2 means that a number is made by taking two different numbers, squaring both, and adding the squares, while x^2-y^2 directs us to form a number by choosing two different numbers, squaring both, and subtracting. Clearly then, such short forms as x^2+y^2 and x^2-y^2 are very compact ways of saying a great deal.

Such a number-form as x^2+ax+b is the *type-form* for numbers to be built up by choosing a number, squaring it, adding the product of it and some second number, and then adding a third number. As x^2+ax+b has *three* terms, it is a *trinomial.* But is made up of *three different* numbers, x, a, and b. Since one of these numbers, x, is squared, the trinomial is called a **quadratic** (square-like) *trinomial.*

The form, x^2+ax+b, is then a type-form for *quadratic trinomials.*

175. Type-Forms Interpreted. Since $x+y$ stands for the sum of any two numbers, if we multiply it by itself we get the *square* of the *sum* of any *two numbers.* Multiplying $x+y$ by $x+y$ gives us

$$x^2+2xy+y^2$$

1. Hence the type-form for the square of the sum of two numbers is $x^2+2xy+y^2$. As a type-form, this $x^2+2xy+y^2$ tells us much.

1. *It tells us that the square of the sum of two different numbers is a trinomial.*

2. *It tells us that two of the three terms of the trinomial are made by squaring the numbers to be added separately.*

3. *It tells us that the remaining term of the trinomial is made by doubling the product of the two numbers that were added to give the original sum.*

4. *It tells us that a short way of getting a square of the sum of two numbers is to square each of the two numbers, to form their product and double it, and then to add the three results.*

CHAPTER XIV

FACTORING

176. The **factors** of a number are the numbers whose product is that number. Factors of a number are the *makers* of the number, *by multiplication.*

177. From the law of the algebraic notation and the meaning of integral exponents, the factors of a monomial are the factors of the coefficient and each letter as many times as there are units in its exponent. Thus,

$$6a^3b^2c = 3\cdot 2\cdot aaa\cdot bb\cdot c = 3\cdot 2aaabbc$$

MONOMIAL FACTORS

Type-form: ax+ay+az

178. Polynomials having a *common factor* in every term are the product of a polynomial and a monomial.

By definition of factors, since $3a(2a-3b)=6a^2-9ab$, $3a$ and $2a-3b$ are the factors of $6a^2-9ab$.

The monomial factor is the *greatest common* factor of the *coefficients* multiplied by the *lowest power* of all the *common letters.*

Thus, $14a^2+21a=7a(2a+3)$ and $18x^4-12x^3=6x^3(3x+2)$.

When the monomial factor is one term of the polynomial, the corresponding term in the polynomial factor is 1. Thus,

$$15x^3+10x^2-5x=5x(3x^2+2x-1)$$

179. Rule.— *Divide the polynomial by the monomial factor and write the divisor and the quotient for the factors.*

Factors may always be *checked* by multiplying them together and comparing the product with the number to be factored.

Exercise 62

Factor the following and check the last four:

1. $5a^3-10a^2$
2. $a^2x^3+a^3x^2$
3. $4a^2b-10ab^3$
4. $6x^2+15x^4$
5. $x^3y^2-x^2y^2$
6. $14ab^2+7a^2b$
7. $6a^3x^3+8a^2x^2-2ax^3y$
8. $3a^3x^3+2a^2x^4-4a^2x^2y^2$
9. $4b^3c^3-8abc^2-4a^2bc^3$
10. $ax^2y^3-3a^3xy^2-a^2x^3y^4$
11. $6a^2c^2+9a^2bc^2d-3ac^2$
12. $a^3b^3c^3-3ab^4c^2+a^2b^3c^2$

COMMON COMPOUND FACTOR

Type-form: ax+ay+bx+by

180. The terms of a polynomial may sometimes be so grouped as to show a common *compound* factor.

Consider $$ax+ay+bx+by$$

The first and second terms of this polynomial contain the common factor a, and the third and fourth terms contain the common factor b. Grouping the terms in this manner and factoring each group, we have:

$$a(x+y)+b(x+y)$$

By the use of parentheses, the polynomial is thus reduced to *two* terms, which are *similar* with reference to the compound factor, $x+y$. Combining the terms according to the rule for addition of terms partly similar, §72, we have:

$$(a+b)(x+y)$$

The first term is not always grouped with the second. It may be grouped with the third term, or the fourth.

Factor $ax+bx+2a+2b$, grouping the first with the third term, and the second term with the fourth. Thus,

$$a(x+2)+b(x+2)$$

Exercise 63

Write the factors of the following and check:

1. $ac-ad+cn-dn$
2. $ax-cy+cx-ay$
3. $ax+2x+ay+2y$
4. $an+bn-ax-bx$
5. $a^3+a^2n+an^2+n^3$
6. $x^2-3y-xy+3x$
7. $a^2-mn-an+am$
8. $a^5+a^3x+a^2x^2+x^3$

181. In the preceding examples, a *positive* monomial factor is taken out of each group. Observe the following:

$$ax+ay-bx-by=(a-b)(x+y)$$
$$ax-ay-bx+by=(a-b)(x-y)$$

Convince yourself that the equations are correct by multiplying $a-b$ by $x+y$ and $a-b$ by $x-y$.

A polynomial cannot be factored in this manner unless the compound factor *is the same* in each group.

To get the same compound factor in each group, $-b$ is taken out of the second group in each of the two examples above.

Exercise 64

Factor the following polynomials and check:

1. $an-bn-ax+bx$
2. $bx-by+y^2-xy$
3. $ax-by+ay-bx$
4. $ab+xy-ay-bx$
5. $n^2-nx+ny-xy$
6. $ax^2-by+axy-bx$
7. $a^3+m^2x-am^2-a^2x$
8. $abx-bc+cn-anx$

182. In some cases the compound factor in one group is like the remaining terms of the polynomial, or like those terms *with their signs changed.* In such examples the monomial factor taken out of one group is $+1$ or -1, as, for example,

$$ax-ay+x-y=(a+1)(x-y)$$
$$ax-ay-x+y=(a-1)(x-y)$$

185. *The square of the sum of two numbers is the square of the first number, plus twice the product of the first and second, plus the square of the second.*

Exercise 67

Give the results of the following:

1. $(b+c)^2$ **2.** $(x+1)(x+1)$ **3.** $(ax+b)^2$

4. If a man lives 8 years, he will be n years old. How old was he 8 years ago?

5. $(a+c)^2$ **6.** $(n+3)(n+3)$ **7.** $(a+by)^2$

8. What will represent the sum of 3 consecutive odd numbers of which s is the smallest?

9. $(b+x)^2$ **10.** $(x+2)(x+2)$ **11.** $(2a+b)^2$

12. A man was x years old a years ago. If he lives, how old will he be in b years?

13. $(x+y)^2$ **14.** $(n+4)(n+4)$ **15.** $(x+3y)^2$

SQUARE OF THE DIFFERENCE OF TWO NUMBERS

Type-form: $a^2-2ab+b^2$

186. Since a and b are *any* two numbers, $(a-b)^2$ is the square of the difference of any two numbers. The square of $a-b$ is found by multiplication to be $a^2-2ab+b^2$, or the square of a, minus twice the product of a and b, plus the square of b, or

$$(a-b)^2=a^2-2ab+b^2$$

187. *The square of the difference of two numbers is the square of the first number, minus twice the product of the first and second, plus the square of the second.*

Exercise 68

Give the results of the following, without multiplying:

1. $(b-c)^2$	2. $(n-1)(n-1)$	3. $(ax-b)^2$
4. $(a-c)^2$	5. $(x-3)(x-3)$	6. $(a-by)^2$
7. $(b-x)^2$	8. $(n-2)(n-2)$	9. $(3x-4)^2$
10. $(b-y)^2$	11. $(x-4)(x-4)$	12. $(x-3y)^2$
13. $(a-x)^2$	14. $(n-6)(n-6)$	15. $(4a-5)^2$

188. An arithmetical number may be squared mentally by considering it to be the sum or the difference of two numbers. Thus,

$$46^2=(40+6)^2=1600+480+36=2116$$
$$46^2=(50-4)^2=2500-400+16=2116$$

Exercise 69

Express the squares of these numbers, first as the sum, then as the difference of two numbers:

1. 38^2	2. 47^2	3. 65^2	4. 54^2	5. 73^2
6. 58^2	7. 64^2	8. 76^2	9. 85^2	10. 95^2

189. A **trinomial** may be squared by grouping two terms to make a binomial of it. Thus,

$$(\overline{a+b}+c)^2=a^2+2ab+b^2+2c(a+b)+c^2, \text{ and}$$
$$(a+\overline{b+c})^2=a^2+2a(b+c)+b^2+2bc+c^2$$

Exercise 70

Give the following squares without actually multiplying:

1. $(\overline{a+b}-c)^2$	2. $(a-\overline{b+c})^2$	3. $(\overline{a-b}+c)^2$
4. $(a-\overline{x+2})^2$	5. $(\overline{a+x}-2)^2$	6. $(a+\overline{x-2})^2$

190. If a number is the product of *equal factors*, one of those factors is called a **root** of the number.

The **square** root of a number is one of the *two* equal factors whose product is the number.

Give the square root of 9; 25; 64; 144; a^2; x^4; $4b^2$; $16x^2y^4$; $100a^4b^6c^8$; $(a+1)^2$; $(x-2)^4$.

The **cube root** of a number is one of the *three* equal factors whose product is the number.

Give the cube root of 8; 27; 64; 125; a^3; b^6; $8x^9$; $216a^3b^6$; $27a^3b^6c^9$; $(a+b)^3$; $(x+y)^6$.

TRINOMIAL SQUARES

Type-form: $x^2 \pm 2xy + y^2$

191. A **trinomial square** is the square of a binomial. Thus,

$$(x+y)^2 = x^2+2xy+y^2$$
$$(x-y)^2 = x^2-2xy+y^2$$

Two terms of every trinomial square are *the squares* of the *two terms* of the binomial.

The *other* term is *twice* the product of the *square roots* of the two squares and may be either *positive* or *negative.*

The factors of a trinomial square are therefore *two like binomials,* and the terms of each factor are the *square roots* of the two squares in the trinomial. Thus,

$$4a^2+9b^2+12ab = (2a+3b)(2a+3b)$$

The two squares are $4a^2$ and $9b^2$, their square roots are $2a$ and $3b$, and $12ab$ is twice the product of these square roots.

The method of factoring a trinomial square is stated as a rule, thus:

192. Rule.— *Find the square roots of the two terms that are squares, connect these roots with the* ***sign of the other term,*** *and write the binomial* ***twice*** *as a factor.*

The two factors of a trinomial square being equal, it is evident that *each* factor is the *square root* of the trinomial.

19. $(a+x)^2-2(a+x)+1$

20. $16x^4+x^6+8x^5$

21. $9a^4b^2+30a^2b+25$

22. $(a+b)^2+4(a+b)+4$

23. $4a^4+4a^2b+b^2$

24. $25a^6c^4+10a^3c^2+1$

25. $-6(x-y)+(x-y)^2+9$

26. $64x^2+32x+4$

27. $121+4a^2b^2-44ab$

PRODUCT OF THE SUM AND DIFFERENCE OF TWO NUMBERS

Type-form: $(a+b)(a-b)$

194. Letting a and b represent *any* two numbers, then $(a+b)(a-b)$ represents the product of their sum and difference. The product is found by multiplication to be a^2-b^2, the difference of their squares, *i.e.*,

$$(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2$$

195. *The product of the sum and difference of two numbers is the difference of their squares.*

Exercise 72

Give the following products:

1. $(a+c)(a-c)$
2. $(x-1)(x+1)$
3. $(2a-2)(2a+2)$
4. $(c+b)(b-c)$
5. $(n-2)(2+n)$
6. $(3x+1)(3x-1)$
7. $(a+x)(a-x)$
8. $(x-3)(3+x)$
9. $(3-2a)(3+2a)$
10. $(b-x)(x+b)$
11. $(4-a)(a+4)$
12. $(2x+4)(2x-4)$

196. The product of two arithmetical numbers may be found by writing them as the sum and difference of the same two numbers. Thus,

$$26\times14=(20+6)(20-6)=400-36=364$$

Exercise 73

Give these products as the products of the sum and the difference of two numbers:

1. 38×22 2. 47×33 3. 54×46
4. 66×54 5. 72×68 6. 83×77

197. Two trinomials may sometimes be grouped so as to represent the sum and difference of two numbers. Thus,

$$(a+b+c)(a+b-c)=(\overline{a+b}+c)(\overline{a+b}-c)$$
$$(a+b-c)(a-b+c)=(a+\overline{b-c})(a-\overline{b-c})$$

Hence, $(a+b-c)(a-b+c)=a^2-b^2+2bc-c^2$

Exercise 74

Give the following products:

1. $(a+x+y)(a+x-y)$ 2. $(a-b+c)(a-b-c)$
3. $(a+x-y)(a-x+y)$ 4. $(a-b+x)(a+b-x)$
5. $(x+y-2)(x-y+2)$ 6. $(a+b+3)(a-b-3)$

DIFFERENCE OF TWO SQUARES

Type-form: a^2-b^2

198. By multiplying the *sum* of two numbers by their *difference*, the *difference of their squares* is obtained, thus:

$$(x+2)(x-2)=x^2-4$$

Since the terms of the product are squares of the corresponding terms of the factors, the terms of the factors are the square roots of the terms of the product, or

$$x^2-1=(x+1)(x-1)$$

199. Rule.— *Write for the factors the sum and the difference of the square roots of the terms of the binomial.*

Exercise 75

Factor the following:

1. a^2-x^2	2. x^2-9	3. $9x^4-y^2$
4. a^6-b^6	5. $1-a^2$	6. $4x^6-y^2$

200. When both terms of the difference-factor are squares, that factor may be resolved into two other factors. Thus,

$$x^8-1=(x^4+1)(x^4-1)=(x^4+1)(x^2+1)(x+1)(x-1)$$

Exercise 76

Factor the following:

1. a^2-x^2	2. x^4-1	3. a^6-9x^4
4. a^4-b^4	5. $1-n^4$	6. $9y^2-16$
7. x^4-y^2	8. x^6-9	9. $25-9y^6$
10. a^2-b^8	11. $4-x^2$	12. $4x^6-49$
13. x^6-y^4	14. x^8-1	15. $64-9x^8$
16. a^8-x^8	17. $1-x^6$	18. $9b^4-81$
19. $9a^2-4$	20. a^2-4	21. $4x^6-25$
22. $9-4x^2$	23. x^8-16	24. $9a^4-4x^2$
25. $9a^6-4$	26. x^8-81	27. $4a^4-b^4c^2$

201. One or both of the squares in this type of example may be the square of a binomial. Thus,

$$(a-b)^2-c^2=(a-b+c)(a-b-c)$$

$$a^2-(b-c)^2=(a+b-c)(a-b+c)$$

and, $$(a-b)^2-(x+y)^2=(a-b+x+y)(a-b-x-y)$$

Exercise 79—Review

Give the factors of the following:

1. $9a^4x^4-y^6$
2. $a^8b^4-c^4d^2$
3. $100x^4-y^6$
4. $(x+y)^2+8(x+y)+16$
5. $81a^6+a^4-18a^5$
6. $9m^2+4n^2+12mn$
7. $(a+x)^2+25-10(a+x)$
8. $16-(a+x)^2$
9. $(a-b)^2-(c-2)^2$
10. $(b+c)^2+16(b+c)+64$
11. $64x^8+16x^7+x^6$
12. $49x^2-42xy^2+9y^4$
13. $-12(x+y)+4(x+y)^2+9$
14. $9a^6-b^4c^4$
15. $a^2b^6-x^8y^4$
16. a^4-144b^4
17. $ax-a+bx-b-2cx+2c$
18. a^2-x^2+2a+1
19. $2xy+9-x^2-y^2$
20. $xy+x-3y^2-3y-2y-2$
21. $a(b-c)-(c-b)$
22. x^3-4x^2+2x-8
23. $a^2(b-c)^2+8a(b-c)+16$
24. a^2-6x-x^2-9
25. $a^2+2bc-b^2-c^2$
26. $(x-y)^2-2ac(x-y)+a^2c^2$
27. $(a-1)^2-y^2$
28. $(a+n)^2-(x-1)^2$
29. $9(a-b)^2+12c(a-b)+4c^2$
30. $25x^4-81$
31. $16a^4-16x^4$
32. $9x^6-81y^4$
33. $a^2+2xy+2ac-x^2-y^2+c^2$
34. x^2-y^2-4x+4
35. $a^2+x^2-c^2+2ax$
36. $a^2+b^2-x^2-y^2-2ab+2xy$
37. $49a^2+14ac+c^2$
38. $25a^4-20a^2b+4b^2$

PRODUCT OF TWO BINOMIALS WITH A COMMON TERM

Type-form: $(x+a)(x+b)$

203. Multiplying $x+a$ by $x+b$ and uniting the two terms that are similar with reference to x, we find:

$$(x+a)(x+b)=x^2+(a+b)x+ab$$

The first term of the product is the *square of* x, the second term is the *product* of x *and the sum of* a *and* b, the third term is the *product* of a and b.

204. Principle.— *The product of two binomials with a common term is the square of the common term, plus the product of the common term and the sum of the unlike terms, plus the product of the unlike terms.*

Exercise 80

Give the following products without multiplying:

1. $(x+2)(x+1)$	2. $(2a+b)^2$	3. $(n-6)(n-5)$
4. $(x-2)(x+1)$	5. $(x-4y)^2$	6. $(a+4)(a-1)$
7. $(x-4)(x-3)$	8. $(a+3b)^2$	9. $(y+9)(y+5)$
10. $(a+6)(a-5)$	11. $(4x-2)^2$	12. $(n-4)(n+3)$
13. $(x+5)(x+4)$	14. $(x+by)^2$	15. $(x+7)(x-4)$
16. $(b-6)(b+3)$	17. $(3a-4)^2$	18. $(n-8)(n-2)$
19. $(x+5)(x-4)$	20. $(a+4x)^2$	21. $(x-7)(x+1)$
22. $(a-9)(a+8)$	23. $(3x-5)^2$	24. $(n+8)(n+6)$
25. $(x-3)(x-2)$	26. $(a+6b)^2$	27. $(x+9)(x-4)$
28. $(a-5)(a-2)$	29. $(4a-4)^2$	30. $(a-7)(a+6)$

205. The product of two arithmetical numbers may sometimes be conveniently found by expressing them as *binomials* with a *common term*. Thus,

$$46\times 36=(40+6)(40-4)=1600+80-24$$
$$57\times 42=(50+7)(50-8)=2500-50-56$$

Exercise 81

Give the following products:

1. 38×23	2. $(40+6)(40+4)$	3. 57×47
4. 64×62	5. $(80-3)(80-2)$	6. 76×62

SPECIAL QUADRATIC TRINOMIALS

Type-form: x^2+ax+b

206. The product of any two *binomials* with a *common term* is represented by the following trinomial:

$$\mathbf{x^2+ax+b}$$

It is evident that the factors of such a trinomial are the *two binomials* of which it is the product.

The first term of each factor is the *square root* of x^2, i.e., x, the second terms are the *two factors of b whose sum is a.*

Similarly,

$$x^2+9x+18=(x+6)(x+3)$$
$$x^2-9x+18=(x-6)(x-3)$$
$$x^2+3x-18=(x+6)(x-3)$$
$$x^2-3x-18=(x-6)(x+3)$$

The factors of b whose sum is a in these four examples are, in order:

$+6$ and $+3$ $\qquad$ $+6$ and -3

-6 and -3 $\qquad$ -6 and $+3$

If the third term of the trinomial is *positive*, the second terms of the *factors* have *like* signs; if the third term is *negative*, the second terms of the *factors* have *unlike* signs.

Exercise 82

Give the factors of the following:

1. $a^2-7a+12$	2. n^2-n-12	3. $x^2-17x+30$
4. $a^2-7a-18$	5. n^2+n-30	6. $x^2+14x+48$
7. $a^2-8a+15$	8. n^2+6n+5	9. $x^2-11x-12$
10. $a^2+9a+20$	11. n^2+n-56	12. $x^2+13x+12$
13. $a^2-a-132$	14. n^2+2n-8	15. $x^2-11x+30$

Exercise 83

Give the factors of the following:

1. $3a^2+a-2$
2. $5x^2-17x+14$
3. $4x^2-x-5$
4. $7a^2-17a-12$
5. $6a^2+a-2$
6. $8x^2-45x-18$
7. $7x^2+x-6$
8. $9a^2+32a-16$
9. $8a^2-a-9$
10. $4x^2-8xy+3y^2$
11. $6x^2+x-5$
12. $6a^2-5ab-6b^2$
13. $2a^2-5a+3$
14. $8x^2+6xy-9y^2$

Exercise 84 — Review of Factoring

Factor the following:

1. $a^2-12a-28$
2. $3x^2-13x+12$
3. $1+6x-72x^2$
4. $2a^2+5ax-3x^2$
5. $a^2+19a+84$
6. $81x^6+16+72x^3$
7. $6x^2+31x+35$
8. $49a^4+98a^2+49$
9. x^4-11x^2-42
10. $12a-12a^2-9a^3$
11. a^4-14a^2+45
12. $18a^2+3ab-45b^2$
13. x^4-21x^2+80
14. $121b^2+88b+16$
15. $3a^2-13a-30$
16. $8x^2+49xy-49y^2$
17. a^2+16c^2-8ac
18. $1-a^2-b^2+2ab$
19. $1-8x+12x^2$
20. $16y^2+16y^4+32y^2$
21. $a^2-9ac+8c^2$
22. $12a^2+31ax+9x^2$
23. $9a^2-8ax-20x^2$
24. $36x^8+25x^2-60x^5$
25. x^8+11x^4-60
26. $6x^2+23xy+21y^2$
27. a^6+15a^3+56
28. $49x^2+70xy+25y^2$

29. $15x^2+4xy-4y^2$

30. $36n^2+25x^2-60nx$

31. $b^4-c^2-2b^2+2cd^2+1-d^4$

32. $8ac-4a^2-4c^2+4$

33. $b^2-a^2-c^2-2ac$

34. $a^2+4c^2-9x^2+6x-1-4ac$

35. $4(a-x)-2(x-a)$

36. $b^3+7b^2-3b-21$

37. $ax-by+z-bx+y+az+x+ay-bz$

38. $a^2b-ac^2-abm+c^2m$

39. $2x^3-4x-3x^2+6$

40. $ax-by-z-bx+y-az+x+ay+bz$

41. $(a^2-b^2-x^2)^2-4b^2x^2$

42. $1-x^2-y^2+2xy$

43. $ab+bx-by-ac-cx+cy+az+xz-yz$

44. $(x^2-y^2+z^2)^2-9x^2y^2$

45. $16-4x^2+12cx-9c^2$

46. $bm+bn-bp-cm-cn+cp+m+n-p$

47. $(m-p^2-x)^2-n^2z^2$

48. $p^3-3pq-p^2q+3q^2$

INCOMPLETE TRINOMIAL SQUARES

Type-form: $x^4+x^2y^2+y^4$

208. Some trinomials and binomials which *may be made trinomial squares* by the addition of a square to them may be resolved into two trinomial factors. For example, consider:

$$9a^4+2a^2b^2+b^4$$

This trinomial would be the square of $4a^2b^2$, if the coefficient of the second term were 6. Proceed thus:

$$\begin{array}{l} 9a^4+2a^2b^2+b^4 \\ \quad\;\; 4a^2b^2 \qquad -4a^2b^2 \\ \hline 9a^4+6a^2b^2+b^4-4a^2b^2=(3a^2+b^2+2ab)(3a^2+b^2-2ab) \end{array}$$

Adding $4a^2b^2-4a^2b^2$, which equals zero, to $9a^4+2a^2b^2+b^4$, the value is not changed, and we then have the *difference of two squares.*

209. When the second term of the trinomial is negative, *two different squares* may in some cases be added, thus:

$$\begin{array}{ll} 4a^4-5a^2b^2+b^4 & \qquad 4a^4-5a^2b^2+b^4 \\ \quad\; a^2b^2 \quad -a^2b^2 & \qquad\quad 9a^2b^2 \quad -9a^2b^2 \\ \hline 4a^4-4a^2b^2+b^4-a^2b^2 & \qquad 4a^4+4a^2b^2+b^4-9a^2b^2 \end{array}$$

The factors of these two results are:

$$(2a^2-b^2+ab)(2a^2-b^2-ab)$$

$$(2a^2+b^2+3ab)(2a^2+b^2-3ab)$$

This would seem to indicate that the expression has *two sets* of prime factors, *but this is impossible.*

We find that each of these factors may be factored by the preceding case, § 207, giving the following factors:

$$(2a-b)(a+b)(2a+b)(a-b)$$

$$(2a+b)(a+b)(2a-b)(a-b)$$

These factors, though arranged differently, are alike, and we conclude that *when two squares can be added* to the expression, it can be resolved into *four binomial factors*, and it is immaterial which of these two squares is added to the expression.

When a *binomial* can be factored by this method, it can generally be resolved into *four* binomial factors.

Exercise 85

Factor:

1. x^4+4
2. $64a^4+1$
3. x^4+4y^4
4. $4x^4-17x^2+16$
5. $36a^4+24a^2x^2+25x^4$
6. $9a^4-34a^2+25$
7. $16x^4-72x^2y^2+49y^4$
8. $x^4-10x^2y^2+9y^4$
9. $81a^4+26a^2b^2+25b^4$
10. $a^4-19a^2b^2+25b^4$
11. $36x^4-88x^2y^2+49y^4$
12. $9a^4-8a^2b^2+16b^4$
13. $64x^4+76x^2y^2+49y^4$
14. $81x^4-40x^2y^2+4y^4$
15. $16a^4-76a^2x^2+25x^4$

SUM OF THE SAME ODD POWERS

Type-form: $x^3 + y^3$

211. The **sum of the same odd powers** of two numbers is the product of a binomial and a polynomial.

The following products may be verified by multiplication:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 + y^6 = x^3 + (y^2)^3 = (x + y^2)(x^2 - xy^2 + y^4)$$
$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$$

The binomial factor is the sum of the same odd roots of the two terms of the binomial.

The terms of the polynomial factor are alternately positive and negative.

Exponents in the polynomial factor decrease and increase ***by the exponents in the binomial factor.***

Exercise 87

Give the factors of the following:

1. $a^3 + 8$ **2.** $n^9 + 64$ **3.** $512 + 64a^3$

4. $a^9 + b^3$ **5.** $x^9 + 27$ **6.** $27 + 343x^6$

7. $1 + b^3$ **8.** $a^5 + 32$ **9.** $729 + 64x^9$

10. $x^6 + y^9$ **11.** $8x^6 + 27$ **12.** $27x^3 + 64y^3$

13. $8 + b^3$ **14.** $a^3 + 125$ **15.** $8a^3 + 125b^3$

16. $a^5 - x^5$ **17.** $x^3 + 216$ **18.** $729x^3 + 8y^3$

19. $x^5 + 1$ **20.** $x^6 - 343$ **21.** $1000a^3 + b^3$

22. $a^7 + b^7$ **23.** $a^3 - 512$ **24.** $8x^3 + 343y^3$

25. $x^7 - 1$ **26.** $x^5 + 243$ **27.** $64a^6 + 27b^6$

212. Some binomials, especially those that are the difference of the same powers, have more than one binomial factor.

The binomial, $a^6 - x^6$, has 5 *binomial* divisors. Show what they are and why they are divisors of $a^6 - x^6$.

213. Summary of Factoring.

I. First take out all monomial factors, and retain their product as one factor of the given expression.

After the monomial factors are removed, next notice the number of terms in the remaining factor.

II. Binomials are factored as:

(*a*) The difference of two squares, thus,

$$a^2-b^2=(a+b)(a-b)$$

(*b*) The difference of the same odd powers,

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4), \text{ etc.}$$

(*c*) The sum of the same odd powers,

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4), \text{ etc.}$$

III. Trinomials are factored as:

(*a*) A trinomial square, thus,

$$a^2\pm 2ab+b^2=(a\pm b)(a\pm b)$$

(*b*) A quadratic trinomial,

x^2+ax+b, or ax^2+bx+c, by *inspection and trial.*

(*c*) A form reduced to II*a*, thus,

$$x^4+x^2y^2+y^4=(x^2+y^2)^2-(xy)^2$$

IV. Polynomials of four or more terms are factored:

(*a*) By grouping terms, thus,

$$ax+bx+ay+by=(a+b)\,x+(a+b)\,y=(a+b)(x+y)$$

(*b*) By II*a*, thus,

$$a^2+b^2+2ab-c^2=(a+b)^2-c^2=(a+b-c)(a+b+c)$$

REVIEW

214. Observe the following rules and factor the exercises below:

I. If the expression contains a monomial factor, that factor should in every case be removed first.

II. All binomials should first be factored as the difference of two squares, if this is possible.

III. Do not write a compound factor as one of the factors of an expression until you are sure it is prime.

Exercise 88 — General Review of Factoring

1. a^4+15a^2+44 **2.** $8a^2+37ab-15b^2$

3. x^4-x **4.** a^4-81 **5.** $x^5 \cdot x^6y+xy$

6. x^6+10x^3-11 **7.** $50a^2-35ab-4b^2$

8. $x^2+2xy+y^2-4x-4y$

9. $9a^2-18ab+8b^2$ **10.** x^6-4x^3-32

11. $ax+cy+x-ay-cx-y$

12. $a^2-m^2+an-mn$ **13.** $20x^2+8x-9$

14. $5a^2+8ab+3b^2+5a+3b$

15. $25a^2-9(3b-2c)^2$ **16.** $18a^2-ax-4x^2$

17. $x^2+y^4-z^4-2z^2-2xy^2-1$

18. x^6+y^6 **19.** $5x^6y^2-5x^3y^5$ **20.** x^3-216

21. $(a+x)^2-1-2x(a+x-1)$

22. $4x^4-61x^2y^4+81y^8$ **23.** $a^5-a^4-a^3+a^2$

24. $a^2-b^2-c^2+2bc+a-b+c$

25. $(a^2+x^2-y^2)^2-4a^2x^2$ **26.** $1+19a^2-20a^4$

27. $a^2+4c+1+b^2-2ab+4c^2$

28. $3(a-b)-(b-a)$ **29.** $3c^3-c^2-3c+1$

30. $3x^2-6xy+3y^2+3xz-3yz$

CHAPTER XV

EQUATIONS. EXERCISES FOR REVIEW AND PRACTICE

SOLUTION OF EQUATIONS BY FACTORING

215. Many equations containing the square of the unknown number may be solved by means of factoring. Thus,

$$7x^2-8=4x^2+19$$

If we transpose all terms to the first member, unite the terms containing x^2, and divide both members of the equation by the coefficient of x^2, we have the following:

$$x^2-9=0$$

Factoring the first member, the result shows the indicated product of two factors equal to 0.

$$(x+3)(x-3)=0$$

A product is 0, if *one of its factors* is 0. Since this product $(x+3)(x-3)$ is 0, at least one factor of it is 0.

If $x+3=0$, then $x=-3$; and if $x-3=0$, then $x=3$. Since both numbers satisfy the equation, $x=+3$ and -3. Both are roots.

It is important to notice here that equations containing the square of the unknown number have two roots.

216. The statement in italics can be made clearer by a little graphing.

I. We begin by showing the graphical solution of

$$x^2-4=0 \qquad (1)$$

We first calculate and make the graph of the first member

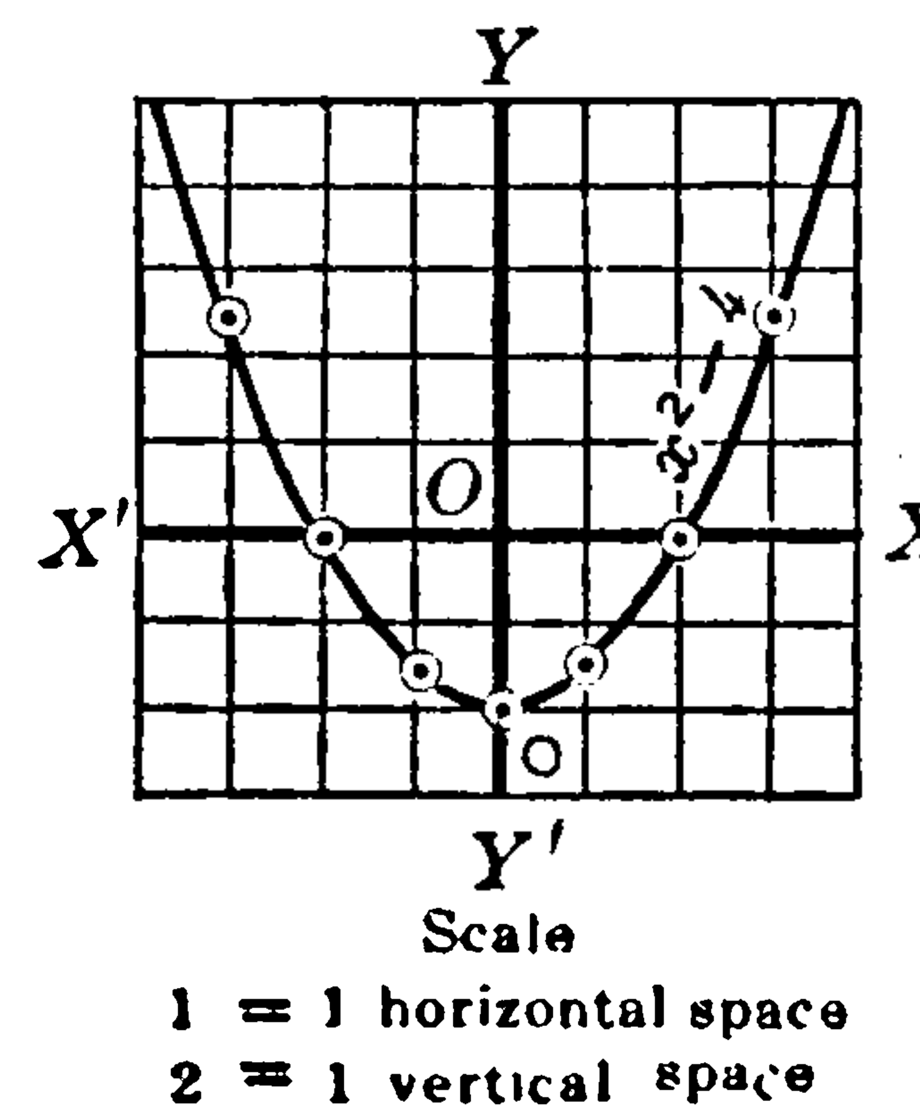
$$x^2-4.$$

Thus calculate:

$$\begin{cases}x=1\\x^2-4=-3\end{cases}\quad\begin{cases}x=2\\x^2-4=0\end{cases}\quad\begin{cases}x=3\\x^2-4=5\end{cases}\quad\begin{cases}x=0\\x^2-4=-4\end{cases}\quad\begin{cases}x=-1\\x^2-4=-3\end{cases}$$

$$\begin{cases}x=-2\\x^2-4=0\end{cases}\quad\begin{cases}x=-3\\x^2-4=+5\end{cases}$$

Graphing the x-values horizontally and the corresponding x^2-4-values vertically, and connecting the points with a smooth curve, we obtain the curve of the figure.



Graph of x^2-4
A Parabola

Equation (1) really asks, "What value or values of x make $x^2-4=0$?" Since the x^2-4-values are the vertical distances from the horizontal to the curve, this question amounts to asking, "What are the x-values where the curve crosses the x-axis?" The answer is seen from the figure to be $x=+2$ and $x=-2$.

Both $+2$ and -2 substituted for x in $x^2-4=0$ satisfy it. For this equation then there are two values of x, because the curve crosses the horizontal in two points.

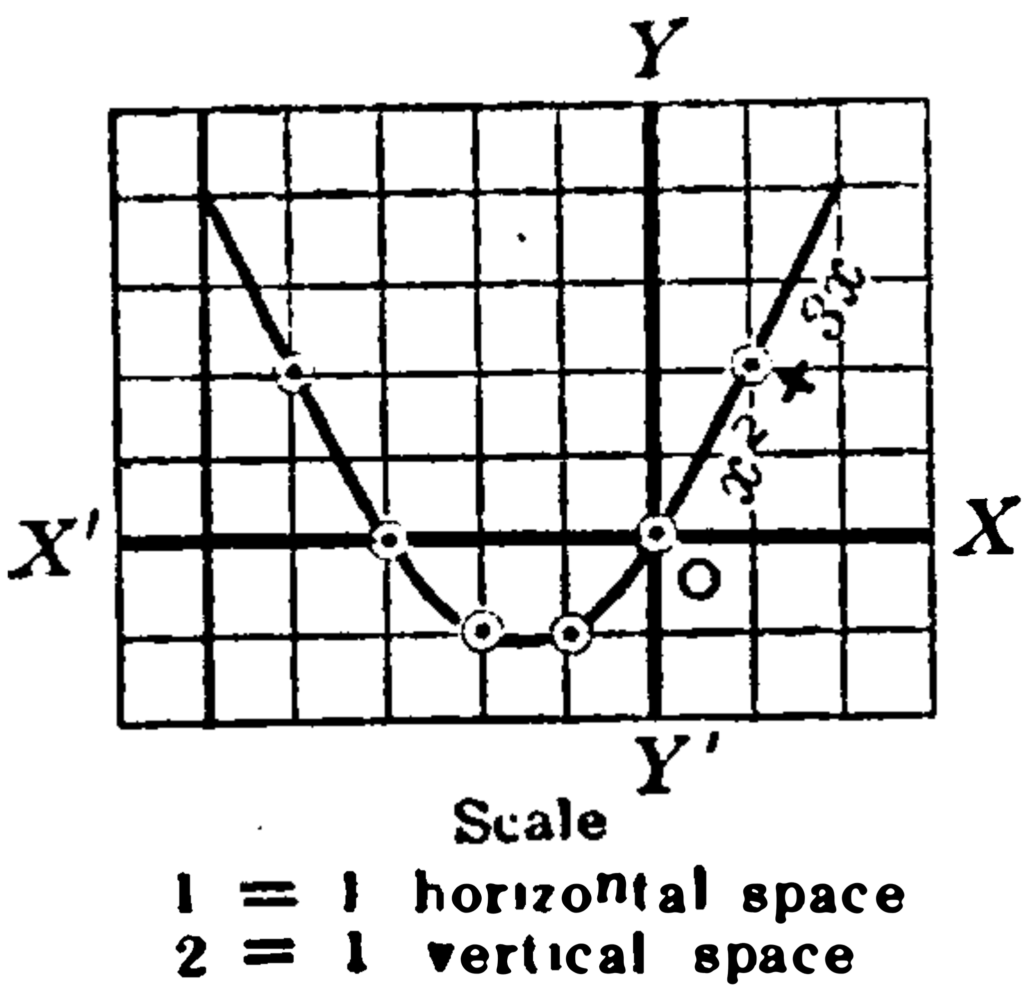
Observe that we graph $f(x)=x^2-4$ and obtain the *parabola*. The solutions of $f(x)=0$ are the x-values of the crossing-points of the parabola over the horizontal axis.

II. Let us now solve graphically the equation,

$$x^2+3x=0 \qquad (2)$$

Graph the first member, x^2+3x, first calculating:

$x=1$,	2,	3,	0,	-1,	-2,	-3,	and -4
$x^2+3x=4$,	10,	18,	0,	-2,	-2,	0,	and $+4$



Graph of x^2+3x
A Parabola

Graphing and connecting the points, we get the graph of x^2+3x as in the figure. Clearly the x-values of the crossing-points, *i.e.*, the points where x^2+3x is equal to 0, are $x=0$ and $x=-3$. These values both satisfy $x^2+3x=0$, and we again have two values because there are two crossing-points.

Here we graph $f(x)=x^2+3x$ obtaining a parabola, whose crossing-points over the horizontal give the x-distances that are the solutions of $f(x)=0$.

III. The graphical solution of the more general form

$$x^2-6x+8=0 \qquad (3)$$

is obtained by first graphing x^2-6x+8.

Calculating the values:

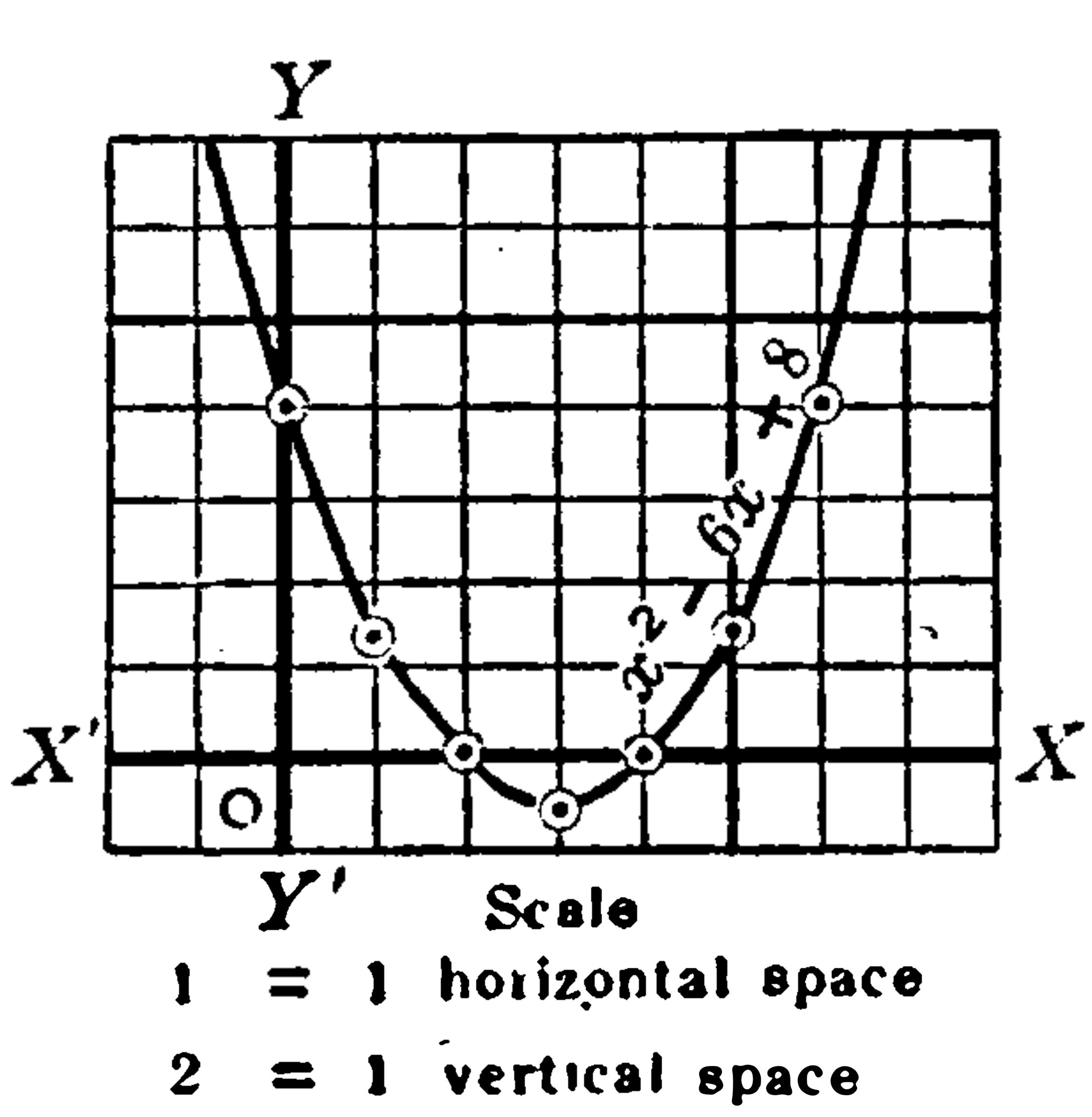
$x=$	1,	2,	3,	4,	5,	6,	0,	-1
$x^2-6x+8=$	3,	0,	-1,	0,	$+3$,	$+8$,	$+8$,	$+15$

The curve is shown in the figure and there are again two crossing-points, $x=+2$ and $x=+4$, and these satisfy

$$x^2-6x+8=0.$$

For *any* equation containing the square of the unknown, the graph of the first member would be such a curve as we have found above.

Hence, *equations containing the square of the unknown have, in general, two roots.*



Graph of x^2-6x+8
A Parabola

The parabola is also the graph of $f(x)=x^2-6x+8$, and its crossing-points over the horizontal give the solutions of $f(x)=0$.

Exercise 90 — Questions and Oral Work

Answer the questions in number symbols and perform the operations indicated in the even numbered exercises.

1. What may represent the area of any rectangle the length of which is twice the width?

2. $(x+1)(x-1)$ $(2x+3y)^2$ $(a+2)(a-1)$

3. How many days will it take a man to build m yards of wall, if he builds n feet a day?

4. $(a-n)(a-n)$ $(3a-4b)^2$ $(x+3)(x+1)$

5. If a man lives x years, he will be y years old. How old was he c years ago?

6. $(x+3)(x-3)$ $(4x+5y)^2$ $(a+5)(a-5)$

7. Express 4 times the product of a square and x cube, increased by n times the number $2x-3y$.

8. $(a-y)(a+y)$ $(2a-ab)^2$ $(x-5)(x-5)$

9. If $2x-5$ represents an odd number, what will represent the next smaller odd number?

10. $(a+7)(a+3)$ $(xy+5x)^2$ $(4+x)(x-4)$

11. Express 7 times the third power of x, diminished by 3 times the sum of $2a$ and $5b$.

12. $(x+3)(x-2)$ $(3a-ab)^2$ $(a-5)(a+4)$

13. A boy has x silver dollars, y dimes, and z 5-cent pieces, \$5.80 in all. What expression equals 580?

14. $(x-2)(x+2)$ $(xy+4y)^2$ $(x-7)(x+3)$

15. If a park is l rods long and w rods wide, how many times must a man walk around it to travel n miles?

16. $(a-n)(a+n)$ $(5a-ab)^2$ $(n-9)(n-5)$

17. If a square is formed by adding 3 feet on all sides of a smaller square, how many square feet are added?

Exercise 91

Simplify the first six of the following equations and solve the rest by factoring:

1. $3x^2-15a^2=2x^2-11a^2$
2. $3y^2+13b^2=4y^2-12b^2$
3. $2x^2+17a^2=5x^2-10a^2$
4. $5y^2-40n^2=3y^2+32n^2$
5. $7x^2-43a^2=3x^2+16a^2$
6. $6y^2+24b^2=8y^2+22b^2$

Some equations that contain both the first and the second powers of the unknown number may be solved by factoring.

$$x^2+x-12=0$$
$$(x+4)(x-3)=0$$
$$x=-4 \text{ and } 3$$

$$4x^2+19x-5=0$$
$$(x+5)(4x-1)=0$$
$$x=-5 \text{ and } \tfrac{1}{4}$$

7. The square of a certain number diminished by 3 times the number is 130. Find the number.

8. $9x^2-6x-12=3x^2$
9. $3y^2-3y-25=2y^2-15$

10. The product of 3 times and 5 times a certain number is 735. Find the number.

11. $8x^2+4x-80=4x^2$
12. $5n^2+14n-6=3n^2-18$

13. If to the square of a number the number itself is added, the sum is 240. Find the number.

14. $5x^2+21x+36=2x^2$
15. $7y^2+5y-45=2y^2-15$

16. The sum of the squares of two consecutive even numbers is 580. Find the numbers.

17. $8x^2+27x+42=5x^2$
18. $8y^2-32y+8=4y^2-52$

19. The quotient of one number divided by another is 4, and their product is 256. Find the numbers.

20. The length of one square field is twice that of another, and both together contain 1280 square rods. What is the length of each side of the smaller square?

EXERCISES FOR REVIEW AND PRACTICE

Exercise 92 — Oral Practice

Answer in number symbols and perform indicated operations:

1. What will represent a number in which there are z hundreds, y tens, and x units?

2. $(a-3)(3+a)$ $\qquad$ $(a^2+b^2)^2$ $\qquad$ $(s-5)(s-4)$

3. If $2x+1$ represents an odd number, what will represent the next smaller odd number?

4. $(x-3)(x+2)$ $\qquad$ $(a^3-b^3)^2$ $\qquad$ $(a+5)(a-3)$

5. A has n cows, B has 5 more than A, and C has as many as A and B together. How many have all?

6. $(4+b)(b-4)$ $\qquad$ $(a^4+b^4)^2$ $\qquad$ $(n+9)(n-8)$

7. What will represent the sum of five consecutive even numbers of which m is the middle one?

8. $(x-7)(x+5)$ $\qquad$ $(a^5-b^5)^2$ $\qquad$ $(a+8)(a+2)$

9. The perimeter of a square is $12x$ feet. What will denote the number of square feet in its area?

10. $(b-5)(5+b)$ $\qquad$ $(3a+2b)^2$ $\qquad$ $(b-9)(b-6)$

11. At m dollars a week for men and b dollars a week for boys, how much will 6 of each earn in 4 weeks?

12. $(x+7)(x+4)$ $\qquad$ $(4x-xy)^2$ $\qquad$ $(y+8)(y-3)$

13. How many square yards are there in the ceiling and walls of a room $4x$ ft. by $3x$ ft. and y ft. high?

14. $(7+x)(x-7)$ $\qquad$ $(5a+3b)^2$ $\qquad$ $(a-8)(5+a)$

15. What may represent the area of any rectangle the length of which is 8 inches greater than its width?

16. At a cents a square yard, what will it cost in dollars to plaster a ceiling l feet long and w feet wide?

Exercise 94 — Oral Practice

Formulate the odd numbered exercises and give the products in the even numbered exercises.

1. If a rectangle is 6 in. longer than wide, what are the dimensions, if each is increased 8 in.?

2. $(x+7)(x+7)$ $\quad$ $(a+3)(a-1)$ $\quad$ $(x+4)(x+2)$

3. What is the area of a square formed by adding 3 feet on all sides of a square x feet long?

4. $(x-6)(x+5)$ $\quad$ $(a-2)(a-1)$ $\quad$ $(x+4)(x-3)$

5. What may represent the perimeters of the first and the enlarged rectangles in the first problem?

6. $(x-7)(x-5)$ $\quad$ $(a-8)(a-8)$ $\quad$ $(x-4)(x+1)$

7. What will represent the sum of four consecutive even numbers of which n is the largest?

8. $(x+6)(x+5)$ $\quad$ $(a-8)(a+3)$ $\quad$ $(x+9)(x+9)$

9. What restriction is placed on the exponents used in proving the law of exponents for multiplication? (§ 132.)

10. $(x-8)(x-6)$ $\quad$ $(a+8)(a-7)$ $\quad$ $(n+8)(n+3)$

11. What does $(x+2)^2$ represent, if x in the expression represents the side of a square?

12. $(s-7)(s-7)$ $\quad$ $(b-9)(b+3)$ $\quad$ $(x-9)(x-4)$

13. Write 5 times the square of $a-b$, diminished by the product of the binomials, $x-7$ and $x-9$.

14. $(n+5)(n+2)$ $\quad$ $(a+8)(a+8)$ $\quad$ $(x+7)(x-1)$

15. What does $(x-4)(x-3)$ represent, if x in the expression represents the side of a square?

16. $(s-9)(s-9)$ $\quad$ $(b-6)(b+4)$ $\quad$ $(y+9)(y+7)$

17. What will represent the quotient of a number of three figures divided by 3 times the sum of the digits?

Exercise 95 — Problems for Review

Solve the following problems and exercises:

1. $\frac{x+4}{x-4}+\frac{x-4}{x+4}=2\frac{1}{2}$

2. $\frac{3(x+4)}{4}=\frac{4x+x^2}{8}$

3. The sum of two numbers is 24, and their product is 128. Find the numbers.

4. The sum of the squares of three consecutive odd numbers is 371. Find the numbers.

5. The sum of two even numbers is 18, and the sum of their squares is 164. Find the numbers.

6. Find two numbers whose difference is 8 and whose sum multiplied by the smaller number is 280.

7. Find two consecutive numbers the sum of whose squares exceeds 10 times the smaller number by 155.

8. Find the side of a square whose area is doubled by increasing its length 6 in. and its width 4 in.

9. The square of the sum of two consecutive numbers exceeds the sum of their squares by 112. Find the numbers.

10. A man worked 17 times as many days as he received dollars per day and earned $272. How many days did he work and how much did he receive per day?

11. At 20¢ a square foot, it cost $56 to lay a parquet floor in a room whose length is 6 feet more than its width. Find the dimensions of the floor.

12. A mason worked 32 days more than he received dollars per day for his labor and earned $105. How many days did he work and how much did he receive per day?

13. An aeroplane flew 50 more miles an hour than the number of hours it flew. It flew 399 miles on the trip in question. How long was it in making the trip?

Exercise 96 — Oral Review

Answer the questions and perform indicated operations:

1. What does $(x+2)(x-2)$ represent, if x in the expression represents the side of a square?

2. $(x-6)(x-2)$ $\quad$ $(5a-5b)^2$ $\quad$ $(n+7)(n+6)$

3. At x cents a rod, how many dollars will it cost to enclose a rectangular field l rods by w rods?

4. $(x+3)(x+2)$ $\quad$ $(a+7)(a-5)$ $\quad$ $(n-4)(n+2)$

5. What is the area of a square formed by cutting off a strip 2 yards wide from all sides of a square x yards long?

6. $(x+9)(x-7)$ $\quad$ $(a-9)(a+5)$ $\quad$ $(n-8)(n-3)$

7. What will represent the quotient of a number of x hundreds, y tens, and z units, divided by 8?

8. $(x+8)(x+4)$ $\quad$ $(a+8)(a-4)$ $\quad$ $(n-9)(n-7)$

9. What is received for x sheep bought at a dollars a head and sold at a profit of b dollars a head?

10. $(x+6)(x+3)$ $\quad$ $(a+5)(a-2)$ $\quad$ $(n-9)(n+4)$

11. A man worked 8 days of n hours each at x cents an hour. He spent b dollars. How much had he left?

12. $(x-3)(x-1)$ $\quad$ $(a-5)(a+2)$ $\quad$ $(n+8)(n+7)$

13. What is received for y horses bought at p dollars a head and sold at a loss of q dollars a head?

14. $(x+6)(x-2)$ $\quad$ $(a-8)(a+1)$ $\quad$ $(n-7)(n-6)$

15. A rectangular field $5x$ rods long has a perimeter of $18x$ rods. What will denote the area in acres?

16. $(x+9)(x+3)$ $\quad$ $(a+9)(a-5)$ $\quad$ $(n-8)(n-1)$

17. If the quotient is represented by q, the divisor by d, and the remainder by r, what is the dividend?

16. Read the sum of $(a+b)(x+y)$ and $(a-b)(x+y)$. Read the sum of $(a+c)(n-1)$ and $(c-a)(n-1)$.

17. Write an expression that represents 5 times the square of the sum of any two numbers.

18. From what law do we obtain the rule for multiplying a polynomial by a monomial?

19. Show that the difference of the squares of two consecutive integers is an odd number.

20. Represent 3 times the sum of the squares of any two numbers multiplied by their difference.

21. Show when the product of several negative numbers is positive and when it is negative.

22. From $4ab-3ac+2bc$ subtract the sum of $3bc+bd-ac$, $3ab-2bd-bc$, and $bd-2ac-ab$.

23. What does a^3+b^3 represent? What does x^2-y^2 represent? What does $2(a+1)(a-1)$ represent?

24. Define *coefficient*; *exponent*; and show the difference in their meaning or signification.

25. What does $2(a+b)^2$ represent? What does $3(a-b)^2$ represent? What does $(a+b)(a-b)$ represent?

26. Subtract $7x-5y+3z$ from $3x-8y+6z$, subtract result from zero, and add to $4x-3y+2z$.

27. Simplify $12a-(2b-c)+4c-(5a+3b)$ and find its value when $a=7$, $b=-3$, $c=-4$.

28. State the *sign law* of multiplication. State the *index law* of multiplication. Prove both laws.

29. Represent 5 times the sum of the squares of any two numbers multiplied by the square of their sum.

30. How much does the square of $70+3$ exceed the product of $(70+3)(70-3)$? Give result without squaring.

31. Subtract the sum of $5m-a-9n$ and $5b+5n+a-4m$ from $a+4b+6m-4n$.

32. Represent the product of any three numbers, the last two of which differ by 2.

33. How is the dividend found, when the divisor, quotient, and remainder are known.

34. State the *sign law* of division. State the *index law* of division. Prove both laws.

35. Without squaring the binomial, give the difference between $(60+4)^2$ and $(60+4)(60-4)$.

36. Find the value of $(a-b)^2+(b-c)^2+(a-b)+2c^2$ when $a=1$, $b=3$, and $c=-4$.

37. Write the product of 51 and 49 by expressing them as the sum and difference of two numbers.

38. How do you determine whether a trinomial of the form of x^2+bx+c is the product of two binomials?

39. Represent 4 times the sum of the cubes of any two numbers multiplied by the sum of their squares.

40. Show that the difference of the squares of two consecutive odd numbers is twice the sum of the numbers.

41. Add $(a+c)+2a(b+c)$, $b(b-c)+a(a+c)-(b+c)$, $(a+c)-(b-c)-a(b+c)$, and $4(b-c)+(a+c)$.

42. From the sum of $2ab-ac+2bc$ and $2ac-bc-3ab$ subtract the sum of $3ac-4bc-ab$ and $2abc-2ab-2ac$.

43. Find the cost of x books at a¢ apiece, $x+5$ books at b¢ apiece, and $x-3$ books at n¢ apiece.

CHAPTER XVI

HIGHEST COMMON FACTOR. LOWEST COMMON MULTIPLE

HIGHEST COMMON FACTOR

217. A **common divisor,** or **common factor,** of two or more numbers is an *exact* divisor of each of them.

Thus, a^2 is a *common* factor of $2a^3$, $3a^4b$, and a^5bc.

218. The **highest common factor** (h. c. f.) of two or more numbers is the *product* of all their common factors. Thus,

$$x^3 \text{ is the h.c.f. of } x^3,\ x^4y, \text{ and } 2x^3y^2z.$$

The term *greatest common divisor* is used in arithmetic, but it is not applicable in algebra. For example, x^3 above may or may not be *greater* than x. Thus, if $x = \frac{1}{2}$, $x^3 = \frac{1}{8}$, and x^3 is therefore *less* than x. In algebra the term *highest common factor* is used. That is, x^3 is *higher* than x (meaning x^1) in the sense that its *exponent* is *higher* than that of x.

HIGHEST COMMON FACTOR OF MONOMIALS

219. The highest common factor (h. c. f.) of two or more monomials may be determined by inspection. Consider:

$$8a^3c^3, \quad 4a^2bc^4, \quad 16a^4b^2c^2, \quad 12a^4c^3$$

The h. c. f. of the coefficients is 4. The highest common *literal* factors are a^2 and c^2. The h. c. f. is $4a^2c^2$.
Observe that the power of each letter in the h. c. f. is the *lowest* power of that letter found in *any* of the monomials.

220. Rule.— *To the h. c. f. of the coefficients, annex the highest power of each letter common to all.*

6. x^2-6x+9, $x^2+2x-15$, and x^3-27

7. $2ax-2a$, $6ax^2-6ax$, and $2abx-2ab$

8. $18ax^2+6a$, $18ax^2-2a$, and $54ax+2a$

9. $27a^3-64$, $9a^2-16$, and $3a^2-28a+32$

10. $4x^2-20x+25$, $8x^3-125$, and $4x^2-25$

11. x^3+27, x^2-9, $8ax^2+24ax$, and x^4-81

12. $24x^4-81x$, $12x^2-18x$, and $48x^3-108x$

13. $a^2+3ab-18b^2$, a^4-27ab^3, and $(a-3b)^2$

14. a^4+a^2-2, $8a^3b-8ab$, and $4-8a^2+4a^4$

15. $a^2-4ab+4b^2$, a^3-8b^3, and $a^2-ab-2b^2$

16. $9x^2-6x+1$, $6x^2+10x-4$, and $9ax^3-ax$

17. $16a^3bc-8a^5bc$, $16-a^8$, $8-a^6$, and $4-a^4$

18. $a^4-4a^2b^2+3b^4$, a^6-b^6, and $b^4-2a^2b^2+a^4$

19. $x^2-10x+16$, $12xy-3x^3y$, and x^2-4x+4

20. $63a^2-36a$, $49a^2-16$, and $16-56a+49a^2$

21. $24x^2+18x-15$, $1-4x+4x^2$, and $8x^3-2x$

22. $x^2+12x+36$, $x^2-2x-48$, and $x^2-3x-54$

23. $9x^2-12ax+4a^2$, and $2ab+2ay-3bx-3xy$

24. $a^2+2ac+c^2$, $a^3-a^2c-ac^2+c^3$, and $4a^2-4c^2$

25. ay^3+ax^3, $5a^2x^2+5a^2xy$, and ax^2+ay^2+2axy

26. $64a-32ax+4ax^2$, $5(x-4)^2$, and $x^2+2x-24$

27. $5a^4+5ab^3$, $a^2x-a^2y-abx+aby$, and a^3-abx

28. $x^3-x^2y-xy^2+y^3$, $(x-y)^4$, and $x^4-2x^2y^2+y^4$

29. $8a^3-8b^3$, $a^3+a^2b-ab^2-b^3$, and $b^2-2ab+a^2$

30. $a^3+a^2c-ac^2-c^3$, $ax-ay-cx+cy$, and a^2-c^2

31. $16x^3+4x^2-2x$, $2x-16x^2+32x^3$, and $16x^3-x$

LOWEST COMMON MULTIPLE

223. A **multiple** of a number is a number that is *exactly divisible* by that number. For example,

$4ab$, $8ac$, and $2ax$ are multiples of $2a$.

224. A **common multiple** of two or more numbers is a number that is exactly divisible by each of them. Thus,

$12a^2bc$ is a common multiple of $2a$, $3b$, and $2c$.

225. The lowest **common multiple** (l. c. m.) of two or more numbers is the product of all their different factors. Thus,

$18a^3$ is the l. c. m. of $3a$, $9a^2$, and $6a^3$.

226. Principles.— *Every multiple of a number contains **all** the factors of that number.*

The lowest common multiple of two or more numbers contains only the factors of all the numbers.

*If two or more numbers have **no common** factor, their lowest common multiple is their **product**.*

LOWEST COMMON MULTIPLE OF MONOMIALS

227. The lowest common multiple of two or more monomials is determined by inspection. Consider:

$$2abc, \quad 6ab^2c, \quad 3a^2b^2c, \quad 4ab^3, \quad a^2bc$$

The l. c. m. of the coefficients is 12. The lowest common multiple of the literal parts is a^2b^3c. Hence $12a^2b^3c$ is the lowest common multiple.

Observe that the exponent of each letter is the highest exponent that letter has in any one of the monomials.

228. Rule.—*To the lowest common multiple of the coefficients, annex all the letters of each monomial, giving each letter the **highest** exponent it has in any monomial.*

Exercise 100

Give the lowest common multiple of the following:

1. $2a^2,\ 3a^3,\ 5a^2b$
2. $4ax^2,\ 2a^2x,\ 5ay^3,\ 10a^3x$
3. $3x^3,\ 6y^2,\ 9x^2y$
4. $9a^3b,\ 4ab^2,\ 3a^2c,\ 12b^3c$
5. $6a^3,\ 5x^2,\ 3a^4x$
6. $4xy^2,\ 3x^3y,\ 5xy^3,\ 15x^2z$
7. $5n^4,\ 2n^5,\ 8b^2n$
8. $8a^4b,\ 5b^2x,\ 4ax^3,\ 16a^3b$
9. $4a^3,\ 5c^2,\ 7a^5b$
10. $5x^4y,\ 7xy^3,\ 2x^5y,\ 14yz^2$

LOWEST COMMON MULTIPLE OF POLYNOMIALS BY FACTORING

229. The lowest common multiple of polynomials is found by resolving them into their prime factors, and finding *the product of all the different factors.* For example:

$$a^2+7a+12=(a+3)(a+4)$$
$$a^2+8a+16=(a+4)(a+4)$$
$$a^2-4a-32=(a+4)(a-8)$$

The l. c. m. is $(a+3)(a-8)(a+4)^2$.

230. Rule.— *Find the product of all the different prime factors of the numbers, taking each factor as many times as it is found in any of the given numbers.*

The factors of the lowest common multiple may often be determined without writing the factors of the expressions. Consider:

$$2x+y, \qquad 2xy-y^2, \qquad 4x^2-y^2$$

The different factors in these expressions are y, $2x+y$, and $2x-y$, and the lowest common multiple is $y(4x^2-y^2)$.

Exercise 101

Find the l. c. m. of each of the following exercises, determining it without writing the factors, as far as possible:

1. x^2-3x-4 and x^2-1
2. $6a-6b$ and $4a^2-4b^2$

28. $8x^3-64$, $4x^2-16$, and $6x-12$

29. $2a+6b$, $3a-9b$, and $3a^2-27b^2$

30. a^2-4, a^2-4a+4, and a^4+2a^3

31. a^2-b^2, $a-b$, $b+a$, and $b-c$

32. $x^2-5ax-24a^2$ and $x^2+8ax+15a^2$

33. x^2-xy, x^3-y^3, and x^2+xy+y^2

34. a^2-1, $2a+2$, $3a-3$, and $5a-5$

35. $a^2-3ab-4b^2$ and $ax-4a+bx-4b$

36. $ac(x-y)$, $2a(x+y)$, and $3c(x+y)$

37. x^2-1, $1-2x+x^2$, and $1+2x+x^2$

38. $20a-5$, $16a^2-1$, $2a$, and $12a^2+3a$

39. $4a^2c-4b^2c$, $2a^2+2ab$, and $3ab-3b^2$

40. $1+2x+x^2$, $1-2x^2+x^4$, and $(1-x)^2$

41. x^4-4, x^4+4x^2+4, and $4-4x^2+x^4$

42. $a^2+8a+16$, a^2-16, and $a^2-8a+16$

43. x^3+2x^2-4x-8 and x^3-2x^2+4x-8

44. x^2+y^2, $xy-y^2$, $xy+y^2$, and x^2+xy

45. $1+x^2+x^4$, $1-x+x^2$, and $1+x+x^2$

46. $12x^2+12$, $2x^2-2$, $8x+8$, and $4x-4$

47. $2a^4(a^4+x^2)$, $5a^3(a^2-x)$, and $3a^2(a^2+x)$

48. $x^3-x^2y+xy^2-y^3$ and $x^3+x^2y-xy^2-y^3$

49. a^2-a-6, $a^2-11a+24$, and $a^2-6a-16$

50. x^2-2x-3, x^2+2x+1, and $9-6x+x^2$

51. $a^3-3a^2-4a+12$, a^2-4, and a^2-a-6

52. $x^2+7x+10$, $x^2-4x-45$, and $x^2-7x-18$

CHAPTER XVII

FRACTIONS

231. An **algebraic fraction** is the indicated division in fractional form of one number by another (see § 7). As examples, observe:

$$\frac{a+b}{a-b} \qquad \frac{x+y}{x^2+2xy+y^2} \qquad \frac{a^3-b^3}{a+b}$$

232. The **numerator** is the number above the line. The **denominator** is the number below the line.

The numerator of a fraction represents the *dividend*, and the denominator represents the *divisor*.

The numerator and denominator of any fraction taken together are called the *terms of the fraction.*

Recall that the dividing line is a symbol of aggregation as well as one of division. See § 152.

233. An **integer, or integral number,** is a number no part of which is a fraction, as 5, 11, 16.

A fraction of anything is defined in arithmetic as one or more of the equal parts of it; but since the terms of an algebraic fraction may be any numbers, positive or negative, integral or fractional, it is quite evident that the arithmetical definition does not accurately describe an algebraic fraction.

The value of any arithmetical fraction is the quotient of the numerator divided by the denominator. This is true of any algebraic fraction, and for this reason it is defined as in § 231 above.

A fraction whose numerator is $a+b$ and whose denominator is $a-b$, is read: $a+b$ *over* $a-b$, or $a+b$ *divided by* $a-b$.

234. The **sign of a fraction** is the sign written before the line that separates the terms.

235. Since a fraction is an indicated division, by the law of signs in division, § 158, the following is true:

$$\frac{+9}{+3}=+\frac{9}{3} \qquad \frac{-9}{-3}=+\frac{9}{3}$$

$$\frac{+9}{-3}=-\frac{9}{3} \qquad \frac{-9}{+3}=-\frac{9}{3}$$

Changing the signs of both numerator and denominator does not change the sign of the fraction.

Changing the sign of either numerator or denominator changes the sign of the fraction.

If either term of a fraction is a polynomial, its sign is changed by *changing the sign of every term.*

$$\frac{a-b}{x-y}=\frac{-a+b}{-x+y}=\frac{b-a}{y-x}$$

236. Two principles are to be observed when the terms of a fraction are expressed by their factors, viz.

1. *Changing the sign of* **one** *factor in numerator* **or** *denominator changes the sign of the fraction.* For:

$$\frac{(a-b)(b-c)}{(x-y)(y-z)}=-\frac{(a-b)(b-c)}{(x-y)(z-y)}=-\frac{(a-b)(c-b)}{(x-y)(y-z)}$$

This is evident, for changing the sign of *one* factor changes the sign of that *term of the fraction.*

2. *Changing the sign of* **two** *factors in numerator* **or** *denominator does* **not** *change the sign of the fraction.* For:

$$\frac{(a-b)(b-c)}{(x-y)(y-z)}=\frac{(a-b)(b-c)}{(y-x)(z-y)}=\frac{(b-a)(c-b)}{(x-y)(y-z)}$$

This is true, for changing the signs of *two* factors *does not* change the sign of that *term of the fraction.*

4. Change $\frac{x+2}{x-2}$ to an equivalent fraction whose denominator is $(x-2)^2$.

5. Change $\frac{-x}{y-x}$ to an equivalent fraction whose denominator is x^2-y^2.

238. A fraction is *in its lowest terms* when the numerator and denominator have no common factor except 1.

To reduce a fraction to its lowest terms, we must remove all factors found in both numerator and denominator.

This is done by canceling the common factors, which is *equivalent to dividing* both numerator and denominator by them, thus,

$$\frac{15a^3x^2}{20a^2x^3}=\frac{3a}{4x} \qquad \frac{x^2-3x+2}{x^2-5x+6}=\frac{(x-2)(x-1)}{(x-2)(x-3)}=\frac{x-1}{x-3}$$

239. Rule.— *Resolve numerator and denominator into their prime factors and cancel (divide out) all factors common to both.*

When the numerator of a fraction is a factor of the denominator, the numerator of the result is 1. For example,

$$\frac{a+x}{a^2-x^2}=\frac{1}{a-x}$$

It is often advisable to change the sign of a factor in one term to make it like a factor in the other. Thus:

$$\frac{(x+7)(x-4)}{5(4-x)}=-\frac{(x+7)(x-4)}{5(x-4)}=-\frac{x+7}{5}. \quad \text{Why?}$$

We change the sign of the factor, $4-x$, in the denominator and also the sign of the fraction, and then cancel the *common factor*.

Exercise 103

Reduce the following fractions to their lowest terms, giving results at sight as far as possible:

1. $\frac{2a^3}{8a^2}$ 2. $\frac{3x^2}{6x^3}$ 3. $\frac{4nx}{8xy}$ 4. $\frac{6a^2}{3a^4}$ 5. $\frac{12x^2y}{16xy^2}$

6. $\frac{3x^3}{9x^4}$ 7. $\frac{2a^4}{6a^5}$ 8. $\frac{6xy}{9yz}$ 9. $\frac{8x^3}{4x^5}$ 10. $\frac{10a^3b}{15ab^3}$

11. $\frac{4a^4}{6a^3}$ 12. $\frac{6x^3}{8x^5}$ 13. $\frac{7ab}{7bc}$ 14. $\frac{9n^5}{6n^4}$ 15. $\frac{14x^4y}{28xy^4}$

Exercise 104

Reduce to lowest terms, giving results at sight as far as possible:

1. $\frac{a^4x+64ax}{a^2-4a+16}$ 2. $\frac{b^2-1}{b^4-1}$ 3. $\frac{x^2-9}{x^2-6x+9}$

4. $\frac{a^2+a}{3b+3ab}$ 5. $\frac{a^2-x^2}{a^6-x^6}$ 6. $\frac{n-1}{n^2-1}$

7. $\frac{9x^2y-3x^3y}{x^2-8x+15}$ 8. $\frac{a-x}{a^3-x^3}$ 9. $\frac{a^2+4-4a}{a^2+6-5a}$

10. $\frac{x^2-1}{5xy+5y}$ 11. $\frac{x^4-y^4}{x^6-y^6}$ 12. $\frac{a^2-1}{a^3-1}$

13. $\frac{x^2-y^2}{x^2-2xy+y^2}$ 14. $\frac{a^2+x^2}{a^6+x^6}$ 15. $\frac{a^2-1}{a^2-3a+2}$

16. $\frac{2a^2-4a}{3ab-6b}$ 17. $\frac{a^2-b^2}{(a+b)^2}$ 18. $\frac{n^2+1}{n^4-1}$

19. $\frac{a^2-b^2}{a^2+2ab+b^2}$ 20. $\frac{x^3-y^3}{(x-y)^2}$ 21. $\frac{n^3-n}{n^2-2n+1}$

22. $\frac{4x^2y^2+4}{5x^3y^3-5}$ 23. $\frac{a^3+x^3}{(a+x)^3}$ 24. $\frac{y^2-1}{y^3-1}$

25. $\frac{ax^2-a^3}{a^2-2ax+x^2}$ 26. $\frac{x^4-y^4}{(x^2-y^2)^2}$ 27. $\frac{a^2+a-6}{a^2+6a+9}$

28. $\frac{9a^3-6ab}{6a^2b-4b^2}$ 29. $\frac{(a+b)^3}{(a^2-b^2)^2}$ 30. $\frac{n^2-1}{n^6-1}$

31. $\frac{a^3-x^3}{a^2-2ax+x^2}$ 32. $\frac{(x^2-y^2)^2}{(x+y)^4}$ 33. $\frac{x^2-16}{x^2+2x-8}$

34. $\frac{2x^3-4x^2y}{3xy^2-6y^3}$ 35. $\frac{(a+x)^2}{(a^2-x^2)^2}$ 36. $\frac{a^4-1}{a^6+1}$

37. $\frac{a^2x-x^3}{a^2+2ax+x^2}$ 38. $\frac{(a-b)^3}{(a^2-b^2)^2}$ 39. $\frac{a^2+2+3a}{a^2+5+6a}$

240. A **mixed number** is a number one part of which is integral and the other part fractional, as

$$\frac{a+b}{x-y}-c+2 \qquad 2a+3x+\frac{a-5}{x-3}$$

241. A **proper fraction** is a fraction which cannot be reduced to a whole or a mixed number, as

$$\frac{x+y}{a+b} \qquad \frac{abc}{xyz} \qquad \frac{x-3}{y-4}$$

242. An **improper fraction** is a fraction which can be reduced to a whole or a mixed number, as

$$\frac{a^4-b^4}{a^2+b^2} \qquad \frac{x^2-5x+9}{x-2} \qquad \frac{x^6-y^6}{x^2-y^2}$$

REDUCTION OF IMPROPER FRACTIONS

243. An improper fraction is reduced to a whole or a mixed number by performing the indicated division. Thus, to reduce $\frac{a^3+x}{a+2}$ to a mixed number, proceed as follows:

$$\begin{array}{l|l}
a^3+x & a+2 \\
\underline{a^3+2a^2} & \overline{a^2-2a+4} \\
\quad -2a^2+x & \\
\quad \underline{-2a^2-4a} & \\
\qquad\qquad 4a+x & \\
\qquad\qquad \underline{4a+8} & \\
\qquad\qquad\quad x-8 &
\end{array}$$

Therefore, $$\frac{a^3+x}{a+2}=a^2-2a+4+\frac{x-8}{a+2}$$

10. $\frac{n^3-1}{n+1}$ 11. $\frac{a^2+4}{a-2}$ 12. $\frac{x^4+y^4}{x+y}$

13. $\frac{a^3-1}{a-1}$ 14. $\frac{n^2+4}{n+2}$ 15. $\frac{x^5-y^5}{x-y}$

16. $\frac{x^3+1}{x+1}$ 17. $\frac{a^3-8}{a-2}$ 18. $\frac{a^5-b^5}{a+b}$

19. $\frac{15x^2+5x-1}{5x}$ 20. $\frac{n^2+3n-10}{n+2}$

21. $\frac{12a^2-4a+5}{2a}$ 22. $\frac{b^2-7b+12}{b-3}$

REDUCTION OF MIXED EXPRESSIONS

245. Mixed expressions are reduced to improper fractions as in arithmetic, except that when the fractional part is *minus*, the numerator of it is *subtracted*. Observe:

$$a-3+\frac{a^2+9}{a+2}\qquad \begin{array}{lr} (a-3)(a+2)= & a^2-a-6 \\ \text{Adding} & a^2\quad +9 \\ \hline & 2a^2-a+3 \end{array}$$

Hence, $$a-3+\frac{a^2+9}{a+2}=\frac{2a^2-a+3}{a+2}$$

Also $$a-x-\frac{a^2+x^2}{a-x}\qquad \begin{array}{lr} (a-x)(a-x)= & a^2-2ax+x^2 \\ \text{Subtracting} & a^2\qquad +x^2 \\ \hline & -2ax \end{array}$$

Hence, $$a-x-\frac{a^2+x^2}{a-x}=\frac{-2ax}{a-x}\text{ or }-\frac{2ax}{a-x}$$

Exercise 106

Reduce to improper fractions:

1. $a+1+\frac{2a+5}{4a}$ 2. $\frac{8a-2x}{3}-2a+3x$

3. $x-3-\dfrac{3x-4}{2x}$

4. $\dfrac{9x+3y}{5}-3x-2y$

5. $a-4+\dfrac{4a+3}{5a}$

6. $5a-3b-\dfrac{8a-4b}{3}$

7. $a+5-\dfrac{7a+4}{a+5}$

8. $\dfrac{7x^2-6y^2}{3x-2y}-2x-3y$

9. $x-2+\dfrac{4x-5}{x-2}$

10. $2a-4x-\dfrac{8a^2-9x^2}{4a+3x}$

11. $a-4+\dfrac{a-16}{a+4}$

12. $\dfrac{6x^2-9y^2}{2x-3y}-3x-4y$

13. $x+6-\dfrac{x-36}{x-6}$

14. $\dfrac{4x^2+9y^2}{3x-2y}-4x+3y$

15. $a+b-\dfrac{a^2+2b^2}{a+b}$

16. $3a-2x-\dfrac{8a^2-7x^2}{2a-3x}$

LOWEST COMMON DENOMINATOR

246. Two or more fractions have a *common denominator* when their denominators are the same numbers.

The **lowest common denominator** (l.c.d.) of two or more fractions is the l.c.m. of their denominators.

Consider: $$\frac{a}{a-x}=\frac{a(a+x)}{(a-x)(a+x)} \text{ and}$$

$$\frac{a}{a+x}=\frac{a(a-x)}{(a+x)(a-x)}$$

247. Rule.— *Find the lowest common multiple of the denominators for the lowest common denominator.*

Divide this denominator by the denominator of each fraction and multiply both terms of the given fraction by the quotient.

Exercise 107

Reduce the following fractions to equivalent fractions having the lowest common denominator:

1. $\frac{3a^2}{3}, \frac{2ax}{2}, \frac{4xy}{6}$

2. $\frac{x+1}{a}, \frac{x-1}{b}, \frac{x^2-1}{c}$

3. $\frac{2ac}{b}, \frac{4x^2}{a}, \frac{5ax}{c}$

4. $\frac{2}{a-1}, \frac{3}{a+1}, \frac{4}{a^2-1}$

5. $\frac{4}{2ab}, \frac{3}{6a^2}, \frac{b}{4ax}$

6. $\frac{a^2+4}{a}, \frac{a+2}{b}, \frac{a-5}{2}$

7. $\frac{3a^2}{3}, \frac{4ab}{x}, \frac{2xy}{c}$

8. $\frac{b}{x^2-9}, \frac{c}{x-3}, \frac{a}{x+3}$

9. $\frac{a}{2bx}, \frac{4}{5a^2}, \frac{3ab}{2}$

10. $\frac{5}{4-x^2}, \frac{3}{2-x}, \frac{4}{2+x}$

11. $\frac{4ax}{5}, \frac{a}{3x^2}, \frac{4bx}{a}$

12. $\frac{x^2-4}{a}, \frac{a-1}{b}, \frac{b-2}{c}$

13. $\frac{3}{3ax}, \frac{a}{6b^2}, \frac{c}{4ab}$

14. $\frac{a^2+4}{a^2-4}, \frac{a-2}{a+2}, \frac{a+2}{a-2}$

ADDITION AND SUBTRACTION OF FRACTIONS

248. *Similar* fractions are added or subtracted by performing those operations upon the *numerators* and writing the result over the common denominator.

We have learned in division that:

$$\frac{a+c+e-n-x}{b}=\frac{a}{b}+\frac{c}{b}+\frac{c}{b}-\frac{n}{b}-\frac{x}{b}$$

Interchanging the members of this equation, we observe the rule for addition and subtraction of fractions.

249. Rule.— *Reduce the fractions to the l.c.d., change the signs of all the terms of numerators of fractions that are pre-*

19. $\frac{5}{2a-b}-\frac{2}{2a+b}$

20. $\frac{x+y}{x}-\frac{x-y}{y}+\frac{x^2-y^2}{xy}$

21. $\frac{2x-3}{x+3}-\frac{2x+3}{x-3}$

22. $\frac{x}{x-y}+\frac{y}{x+y}-\frac{y^2}{x^2-y^2}$

23. $\frac{3a+b}{a-b}+\frac{a}{3a+b}$

24. $\frac{5}{x+4}+\frac{3}{x-4}-\frac{9x}{x^2-16}$

25. $\frac{4a-1}{2a+2}+\frac{6a+2}{3+3a}$

26. $\frac{a}{a-b}-\frac{b}{a+b}-\frac{b^2}{b^2-a^2}$

27. $\frac{x+y}{y}-\frac{x}{x+y}-\frac{x(x^2-y)}{y(x^2-y^2)}$

28. $\frac{b}{a+b}-\frac{b^2}{(a+b)^2}-\frac{a^2b}{(a+b)^3}$

29. $\frac{4ax}{(a-x)^3}+\frac{4}{a-x}+\frac{4x}{(a-x)^2}$

30. $\frac{2}{(x-y)^2}+\frac{4}{y^2-x^2}+\frac{2}{(x+y)^2}$

31. $\frac{3}{a(a-2)}-\frac{4}{a^2-4}+\frac{2}{a(a+2)}$

32. $\frac{4ab}{b^3-a^3}+\frac{2}{a-b}-\frac{a+b}{a^2+ab+b^2}$

33. $\frac{n+4}{n-5}-\frac{n^2+31}{n^2-2n-15}+\frac{n-2}{n+3}$

34. $\frac{3x^2}{y^3+x^3}-\frac{1}{x+y}+\frac{x-y}{x^2-xy+y^2}$

35. $\frac{x-y}{x-z}-\frac{(x-z)^2}{(x-z)(x-y)}+\frac{x-z}{x-y}$

36. $\frac{a+4}{a^2+a+1}-\frac{a^2+4a-2}{1-a^3}-\frac{1}{a-1}$

37. $\frac{2}{n+1}-\frac{3n+6}{n^2-n+1}+\frac{n^2+3n+5}{n^3+1}$

38. $\frac{a^2+4a+9}{a^3+27}-\frac{2}{a+3}+\frac{a+4}{a^2-3a+9}$

39. $\frac{n^2+2n+28}{8-n^3}+\frac{3}{n-2}-\frac{n+6}{n^2+2n+4}$

40. $\frac{4xy}{x^3+8y^3}+\frac{1}{x+2y}+\frac{x-2y}{x^2-2xy+4y^2}$

MULTIPLICATION OF FRACTIONS

250. The *product* of two fractions is the *product of the numerators* over the *product of the denominators.*

From $\frac{a}{b}=m$ and $\frac{c}{d}=n$, we have $\frac{a}{b}\times\frac{c}{d}=mn.$ Why?

From the first equations, $a=bm$ and $c=dn$. Multiplying $a=bm$ by $c=dn$, member by member, we have

$$ac=bdmn.$$

Dividing both members of $ac=bdmn$ by bd, we have

$\frac{ac}{bd}=mn$, and by the comparison axiom,

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

This method is applicable also when either factor is integral, for integers may be expressed in fractional form.

Since the product of the numerators is divided by the product of the denominators, cancellation may be employed.

Exercise 109

Simplify the following:

1. $\frac{3a}{4x}\times\frac{2b}{5a}\times\frac{5ac}{6ab}$

2. $\frac{a^2-25}{a^2-3a}\times\frac{a^2-9}{a^2+5a}$

3. $\frac{8b}{9a}\times\frac{3a}{2x}\times\frac{3cy}{4ab}$

4. $\frac{a-1}{2a+8a^2}\times\frac{4a^2+a}{a-2}$

5. $\dfrac{x-1}{1-y^2}\times\dfrac{1+y}{x-1}$

6. $\dfrac{x^2-4y^2}{x^2-xy}\times\dfrac{x-y}{x^2+2xy}$

7. $\dfrac{x^2-1}{x^2-4}\times\dfrac{x+2}{x-1}$

8. $\dfrac{3xz+x^2}{y^2+xy}\times\dfrac{x^2-y^2}{x^2-9z^2}$

9. $\dfrac{x^2-y^2}{x^2+xz}\times\dfrac{x+z}{x-y}$

10. $\dfrac{x^3+8y^3}{(x^2-y^2)^2}\times\dfrac{(x+y)^2}{x+2y}$

11. $\dfrac{a^2-b^2}{(a+b)^2}\times\dfrac{a+b}{a-b}$

12. $\dfrac{2y+z}{(x+5)^2}\times\dfrac{x^2-25}{yz^3+8y^4}$

13. $\dfrac{bc+bx}{a^2+ax}\times\dfrac{3ax}{4by}\times\dfrac{c-x}{a-x}\times\dfrac{a^2-x^2}{c^2-x^2}$

14. $\dfrac{(a^2-x^2)^2}{6a^2}\times\dfrac{10b}{a-x}\times\dfrac{3ax}{5by}\times\dfrac{4}{(a+x)^2}$

15. $\dfrac{a+b}{(a-b)^2}\times\dfrac{3ac}{b}\times\dfrac{a-b}{(a+b)^2}\times\dfrac{ab-b^2}{ab+a^2}$

16. $\dfrac{(x-y)^2}{3+x}\times\dfrac{x-4}{x^2-y^2}\times\dfrac{(y+x)^2}{x^3-64}\times\dfrac{x^3+27}{x^2-y^2}$

17. $\dfrac{a^2+ab}{(a-b)^2}\times\dfrac{ab-b^2}{(a+b)^2}\times\dfrac{4x+x^2}{a+x}\times\dfrac{a^2-b^2}{ab^2x}$

18. $\dfrac{a^2-5a+6}{a^2-2a-8}\times\dfrac{a^2+2a}{a^2-4a+4}\times\dfrac{a^2+8-6a}{a^2+3-4a}$

19. $\dfrac{n^2+2n+1}{n^3-n^2-2n}\times\dfrac{n-5}{n^2+4+2n}\times\dfrac{n^4-8n}{n^2-4n-5}$

20. $\dfrac{ax+a^2}{4a-6x}\times\dfrac{a^2+x^2-ax}{2a^3-3a^2x}\times\dfrac{3ab}{2xy}\times\dfrac{4a^2-9x^2}{x^3+a^3}$

21. $\left(\dfrac{1}{b}-\dfrac{1}{a}\right)\times\left(b-\dfrac{b^2}{a}\right)$

22. $\left(1+\dfrac{3a}{1-a}\right)\left(1+\dfrac{a}{1+a}\right)$

23. $\left(\dfrac{x}{y}+\dfrac{y}{x}\right)\times\left(x-\dfrac{y^2}{x}\right)$

24. $\left(1+\dfrac{2x}{1-x}\right)\left(1-\dfrac{2x}{1+x}\right)$

25. $\left(x-\dfrac{y^2}{x}\right)\times\left(x-\dfrac{x^2}{y}\right)$

26. $\left(2+\dfrac{2y}{x-y}\right)\left(1-\dfrac{x-y}{x+y}\right)$

3. $\frac{a+3}{b+1} \div \frac{a^2-9}{b^2-1}$

4. $\frac{a+x}{a^2-3ax} \div \frac{a^2+ax}{a^2-9x^2}$

5. $\frac{x+y}{a+x} \div \frac{x^2+xy}{a^2-x^2}$

6. $\frac{a^2-b^2}{a^2-4b^2} \div \frac{b^2+ab}{2ab+a^2}$

7. $\frac{a-b}{a+b} \div \frac{(a-b)^2}{a^2-b^2}$

8. $\frac{(a-x)^2}{a-2x} \div \frac{(a^2-x^2)^2}{a^3-8x^3}$

9. $\frac{a^2-25}{a^2+6a} \div \frac{a^2-5a}{a^2-36}$

10. $\frac{(x^2-25)^2}{8a^4+ax^3} \div \frac{(x-5)^2}{x+2a}$

11. $\left(\frac{x}{y}-\frac{2}{y}\right) \div \left(x-\frac{4}{x}\right)$

12. $\frac{x^2+4-2x}{x-2} \div \frac{x^2+8}{x^2-4}$

13. $\left(\frac{a}{2}-\frac{2}{a}\right) \div \left(1+\frac{2}{a}\right)$

14. $\frac{a^2-x^2}{x^2-3x-4} \div \frac{a-x}{x^2+x}$

15. $\left(\frac{1}{y}-\frac{1}{x}\right) \div \left(\frac{x}{y}-1\right)$

16. $\frac{n^2-6-n}{n^2-2-n} \div \frac{2n+n^2}{n^2-2n}$

17. $\left(1+\frac{3}{x}\right) \div \left(1+\frac{x}{3}\right)$

18. $\frac{a^3+x^3}{a^2+6a+8} \div \frac{a+x}{4+2a}$

19. $\left(x-\frac{y^2}{x}\right) \div \left(1+\frac{x}{y}\right)$

20. $\frac{a^3-x^3}{a^3+x^3} \div \frac{a-x}{a^2-ax+x^2}$

21. $\left(1-\frac{x^2}{a^2}\right) \div \left(1+\frac{a}{x}\right)$

22. $\frac{x^2+2xy+y^2}{xy+y^2} \div \frac{x^2-y^2}{y^2}$

23. $\left(a-\frac{x^2}{a}\right) \div \left(1+\frac{x}{a}\right)$

24. $\frac{a^2-2+a}{a^2-9} \div \frac{a^2-6-x}{a+3}$

25. $\left(\frac{1}{y^2}-\frac{1}{x^2}\right) \div \left(\frac{y}{x}+1\right)$

26. $\frac{x^3-y^3}{x^2+2xy+y^2} \div \frac{x^2-xy}{x+y}$

27. $\left(x^2-\frac{y^2}{x}\right) \div \left(1-\frac{y}{x}\right)$

28. $\frac{a^2-a-6}{a^2-25} \div \frac{a^2+a-12}{5a+a^2}$

29. $\left(y^2+\frac{x^3}{y}\right) \div \left(1+\frac{x}{y}\right)$

30. $\frac{a^2+6a-7}{2a+b} \div \frac{a^2+a-42}{ab^2+2a^2b}$

31. $\left(\frac{x}{y^2}-\frac{y}{x^2}\right)\div\left(1-\frac{y}{x}\right)$ **32.** $\left(4-\frac{6}{a+1}\right)\div\left(8+\frac{8-2a}{a^2-1}\right)$

33. $\left(1-\frac{x^2}{a^2}\right)\div\left(\frac{a^4}{x^4}-1\right)$ **34.** $\left(a+\frac{8a}{a^2-9}\right)\div\left(a-\frac{-2a}{a-3}\right)$

Exercise 111 — Test Questions and Review

Answer all you can orally:

1. Show that a common divisor of two numbers is a divisor of any multiple of either of them.

2. Write an algebraic expression that is exactly divisible by $a+b$ and $2a-3b$.

3. Give some algebraic expressions of which the lowest common multiple is $2a^4-2x^4$.

4. How do you determine whether a binomial is the product of the sum and difference of two numbers?

5. Show that a common divisor of two numbers is a divisor of their sum and also of their difference.

6. Recalling the solution of equations by factoring, what are the roots of the equation, $x^2-5x+6=0$?

7. Show how much the square of the sum of two numbers exceeds the product of their sum and difference.

8. Without squaring the binomial, give the difference between $(30+7)^2$ and $(30+7)(30-7)$.

9. How much does the square of $40+5$ exceed the product of $(40+5)(40-5)$?

Find the value of the following expressions when $a=1$, $b=2$, $c=3$, $d=4$, $e=0$, $m=\frac{1}{2}$, $n=\frac{1}{3}$.

10. $cd^2m-8a^2b^3\div d^2m^2+9c^2\times 2b^2+6a^3dm^2n-b^2c^2d^2n$

11. $m^3\times d^3+7bcm+9a^3c^3-a^2c^2e^2+c^2d^2\div b^4m+a^5\times d^3$

Find the value of the following expressions when $a=\frac{1}{4}$, $b=\frac{1}{3}$, $c=1$, $d=4$, $x=\frac{1}{5}$, $y=3$.

12. $6a^2d^3-5c^3y^2\times 3b^2x$

13. $8b^3y^3+a(2d^2-2d)+5x^2$

14. $9b^2d^2+8a^2d^3\div 5c^2x^2$

15. $2c^3d^3-(2y^4-5d)x-4y^2$

16. Show how much the square of the sum of two numbers exceeds the square of their difference.

17. Give the difference between $(20+4)^2$ and $(20-4)^2$ without squaring either binomial.

18. What is meant when it is said that a certain number satisfies an equation?

19. By what must a fraction be multiplied to give the smallest possible integral product?

20. Show to what the sum of any two numbers divided by the sum of their reciprocals is equal.

21. Find the h. c. f. and the l. c. m. of $x^4+x^2y^2+y^4$, x^2+xy+y^2, and x^2-xy+y^2.

22. How much does the square of $50+4$ exceed the square of $50-4$? Give result without squaring.

23. What is the result in multiplication of fractions, when all factors in numerator and denominator cancel?

Find the value of the following expressions when $a=2$, $b=1$, $c=4$, $d=3$, $n=5$, $x=\frac{1}{3}$, $y=\frac{1}{2}$.

24. $\dfrac{a-y}{d+n}$

25. $\dfrac{a^2-b}{b^2+y}$

26. $\dfrac{4b^2}{y^2}-\dfrac{2d^2}{9x^2}$

27. $\dfrac{c+y}{c-b}$

28. $\dfrac{c^2+2}{a+y^2}$

29. $\dfrac{bc^2}{y^3}-\dfrac{8n^2}{5b^2}$

30. $\dfrac{2}{x}-\dfrac{2}{y}$

31. $\dfrac{b^2-x^2}{c^2-a^2}$

32. $\dfrac{b^2c}{ax^3}+\dfrac{2c^2}{a^2b}$

CHAPTER XVIII

LITERAL AND FRACTIONAL EQUATIONS. SOLUTION OF FORMULAS

LITERAL AND FRACTIONAL EQUATIONS

254. A **literal equation** is an equation in which there are two or more general numbers.

In solving such equations, the value of *any* letter may be found, but only in terms of the other letters.

Solve for x, $\qquad ax-a^2=bx-b^2$

Adding a^2 and $-bx$ to both members of this equation, and uniting the terms containing x, we have

$$(a-b)x=a^2-b^2$$

By the division axiom, $\qquad x=a+b$

Checking: $\quad a(a+b)-a^2=b(a+b)-b^2$ or $ab=ab$

To solve a literal equation for any letter in it is to find the value of that letter in terms of the others.

Exercise 112

Solve the following equations in the left column for x, and those in the right column for y and check:

1. $4a-x=4b-bx$ $\qquad$ **2.** $2b+6y=3c+ay$

3. $5n-x=4n+nx$ $\qquad$ **4.** $ay-ab=3y-3b$

5. $3a-x=2a-bx$ $\qquad$ **6.** $5a-by=ay-5b$

7. $n^3-nx=ax-a^3$ $\qquad$ **8.** $2a-9y=ay-18$

9. $\frac{2x}{a}-4a=4b-\frac{2x}{b}$ $\qquad$ **10.** $\frac{n}{3}-n+\frac{y}{5}=y-\frac{n}{5}-\frac{y}{3}$

255. Special Devices. It will be well to note here some special devices for clearing equations of fractions. Thus, clear of fractions $\frac{2x-9}{5}+\frac{2x+6}{3x-1}=\frac{4x-8}{10}$.

Multiplying both members by 10, the lowest common multiple of the monomial denominators, we have

$$4x-18+\frac{20x+60}{3x-1}=4x-8$$

Subtract $4x$ in each member, unite other monomials, clear of fractions, and complete the solution.

In some examples it simplifies the solution to combine fractions before clearing of fractions. Thus, from

$$\frac{1}{a-c}+\frac{a-c}{x}=\frac{1}{a+c}+\frac{a+c}{x} \qquad (1)$$

we obtain, $\frac{1}{a-c}-\frac{1}{a+c}=\frac{a+c}{x}-\frac{a-c}{x}$.

Combining the fractions in each member, we have

$$\frac{2c}{a^2-c^2}=\frac{2c}{x}$$

If two equal fractions have the same numerator, not 0, their denominators are equal. Hence,

$$x=a^2-c^2$$

Check by substituting in equation (1)

Exercise 113

Solve the following equations:

1. $\frac{3x+8}{12}=\frac{4x-3}{3x+4}+\frac{x}{4}$

2. $\frac{4y+8}{y^2-4}-\frac{y-2}{2+y}=\frac{y+2}{y-2}$

3. $\frac{2x-5}{3x-2}+\frac{x}{3}=\frac{5x+8}{15}$

4. $\frac{3y+4}{1-y^2}-\frac{y+3}{1-y}=\frac{1-y}{y+1}$

5. $\frac{5x-4}{10}=\frac{3x+8}{2x+5}+\frac{x}{2}$

6. $\frac{a+1}{a-1}=\frac{5x-a}{a^2-1}+\frac{a-1}{a+1}$

7. $\frac{2x}{a}=x-a+\frac{4}{a}$

8. $\frac{3x}{10}+\frac{x}{5}=50-\frac{x}{3}$

9. $\frac{3x}{4}-\frac{2x}{5}=x+3$

10. $\frac{x-c}{x-a}=\frac{(2x-c)^2}{(2x-a)^2}$

11. $\frac{2+\frac{4cx}{ab}}{cx}=\frac{\frac{4ab}{cx}+2}{ab}$

12. $\frac{a+c}{ac}=\frac{x-2a}{a}+\frac{x}{c}$

13. $\frac{5x+4}{10}-\frac{5x-4}{2x+2}=\frac{x}{2}$

14. $\frac{2x-a}{2x+a}+\frac{2b-x}{4b+x}=0$

15. $\frac{2x+1}{5}-\frac{3x-2}{6x+3}=\frac{6x-1}{15}$

16. $\frac{x-2}{c}+\frac{2x}{a}=5+\frac{6c}{a}$

17. $\frac{3x+2}{4}=\frac{5x+6}{x-6}+\frac{x}{3}$

18. $\frac{x-2ac}{bx}-\frac{1}{x}=\frac{x-3b}{acx}$

19. $\frac{1}{x-5}+\frac{4}{x+1}=\frac{5}{x-2}$

20. $\frac{8x+3}{x^2-9}-\frac{x+2}{3+x}=\frac{x+3}{x-3}$

21. $\frac{5x+4}{6}-\frac{6x+4}{3x+2}=\frac{x+6}{5}$

22. $\frac{6}{x+2}-\frac{9}{2x+4}=\frac{8}{8x+3}$

23. $\frac{3x+8}{6}-\frac{4x+8}{3x+6}=\frac{2x+5}{4}$

24. $\frac{2-x}{x+2}=\frac{2x^2+2x}{4-x^2}-\frac{x+2}{2-x}$

25. $\frac{2x+1}{2x-1}-\frac{3x^2}{4x^2-1}=\frac{2x-1}{1+2x}$

26. $\frac{12x+11}{8}-\frac{9x+7}{6}=\frac{10x-5}{46x+8}$

27. $\frac{x+4a+c}{x+a+c}+\frac{4x+a+2c}{x+a-c}=5$

28. $\frac{2}{x-2}-\frac{3}{x-3}=\frac{4}{x-4}-\frac{5}{x-5}$

29. $\frac{1}{(x-7)(x+2)}=\frac{1}{(x-4)(x-3)}$

Exercise 114

Solve the following equations:

1. $\dfrac{x+2}{x-2}+\dfrac{x+4}{x-3}=\dfrac{2x^2-3x+2}{x^2-5x+6}$

2. $\dfrac{x-4}{x-5}+\dfrac{x-8}{x-9}=\dfrac{x-7}{x-8}+\dfrac{x-5}{x-6}$

3. $\dfrac{x+1}{x-1}+\dfrac{x+4}{x-4}=\dfrac{x+2}{x-2}+\dfrac{x+3}{x-3}$

4. $\dfrac{x+5}{x+4}-\dfrac{x-6}{x-7}=\dfrac{x-4}{x-5}-\dfrac{x-15}{x-16}$

5. $\dfrac{a^2+x}{c^2-x}-\dfrac{a^2-x}{c^2+x}=\dfrac{4acx+2a^2-2c^2}{c^4-x^2}$

6. $\dfrac{x+3a}{x+a}+\dfrac{x+2a}{x-a}=\dfrac{x+a}{x+3a}+\dfrac{x+2a}{x+5a}$

7. $\dfrac{5x-8}{x-2}+\dfrac{6x-44}{x-7}-\dfrac{10x-8}{x-1}=\dfrac{x-8}{x-6}$

8. $\dfrac{5x-64}{x-13}+\dfrac{x-6}{x-7}=\dfrac{4x-55}{x-14}+\dfrac{2x-11}{x-6}$

9. $\dfrac{9x+4}{15}-\dfrac{3x+2}{3x-4}+2\frac{1}{3}=\dfrac{3x+3}{5}-\dfrac{2x-5}{3x-4}$

10. $\dfrac{8x+5}{9}-\dfrac{4}{5}x-\dfrac{3x-a}{2x-a}=\dfrac{4x-2}{45}-\dfrac{2x-2a}{2x-a}$

11. $\dfrac{5}{2(y-1)}-y+\dfrac{y^2}{y-1}=\dfrac{3}{y-1}-4+\dfrac{4}{3(y-1)}$

12. $\dfrac{y}{y-3}-\dfrac{2y^2}{3(y-3)}=\dfrac{y-2}{3}-6-\dfrac{9}{3(y-3)}-y$

13. $\dfrac{y-2}{2+y}-\dfrac{9y-1}{2(y-2)}-2\frac{1}{2}=\dfrac{2+y}{y-2}-\dfrac{7y+86}{2(y^2-4)}-7$

14. $\frac{x+2}{x}+\frac{x-7}{x-5}-\frac{x+3}{x+1}=\frac{x-6}{x-4}$

15. $\frac{x+3}{x+1}+\frac{x-6}{x-4}=\frac{x+4}{x+2}+\frac{x-5}{x-3}$

16. $\frac{x-5}{x+5}-\frac{x-4}{x+4}=\frac{x-10}{x+10}-\frac{x-9}{x+9}$

17. $\frac{2c}{x+4c}+5=\frac{2x+3c}{x+c}+\frac{3x+6c}{x+2c}$

Exercise 115 — Problems in Simple Equations

Solve the following problems:

1. Separate 59 into two such parts that 4 times the smaller shall exceed twice the larger by 26.

2. From what number must 135 be subtracted to get 273?

3. Find the number to which if 329 be added, the sum will be 642.

4. What number must be multiplied by 37 to obtain 999?

5. A is 3 times as old as B, but in 20 years he will be only twice as old. Find the age of each.

6. What number must one divide by 23 to obtain 163?

7. Divide 220 so that the quotient of one part divided by the other is 4 and the remainder 20.

8. What number must be added to .378 to give .65?

9. A is 53 years old and B is 33. How many years have elapsed since A was $1\frac{4}{5}$ times as old as B?

10. What number must one subtract from $3\frac{2}{3}$ to get $2\frac{1}{2}$?

11. Divide \$15 into two parts so that there are twice as many dimes in the first part as there are 5-cent pieces in the second part.

12. By what number must one multiply $3\frac{1}{3}$ to obtain $7\frac{1}{7}$?

13. The difference between two numbers is 17; and if 4 is added to the larger number, the sum is 4 times the smaller number. Find the numbers.

14. Divide \$9000 into two parts such that the interest on the greater part for 2 years at 6% shall be equal to the interest on the other part for 3 years at 5%.

15. What number subtracted from 164 gives the same result as 92 added to the number?

16. Of what number is $5\frac{1}{5}$ the three-tenths part?

17. The difference between two numbers is 32; and if the greater is divided by the less, the quotient is 5 and the remainder 4. Find the numbers.

18. By what number must one divide $3\frac{1}{3}$ to get $5\frac{1}{5}$?

19. Three men earned a certain sum of money. A and B earned \$180; A and C earned \$190; and B and C earned \$200. How much did they all earn?

20. What number is as much under $7\frac{1}{3}$ as it is over $5\frac{1}{5}$?

21. The length of a rectangle is $1\frac{2}{3}$ times its width. If each dimension were 3 inches less, the area would be diminished 279 square inches. Find the length.

22. What number lies midway between $3\frac{1}{4}$ and $7\frac{1}{8}$?

23. A man bought a coat for \$36, paying for it in 2-dollar bills and 50-cent pieces, giving twice as many bills as coins. How many bills did he give?

24. Of what number does the double exceed by 9 its $\frac{1}{5}$?

25. A man invested a certain sum at 5% and twice as much at 6%. His annual income from both investments was \$765. How much did he invest?

35. A can do a piece of work in 12 days, which B can do in 18 days, and with C's help they can do it in 4 days. In how many days can C do the work?

36. If A can do half of a piece of work in 10 days and B can do the whole of it in 15 days, in how many days can both of them do it working together?

37. A speculator bought two pieces of land at the same price. He sold one piece at a profit of $1700 and the other at a loss of $900, receiving twice as much for one piece as for the other. How much did each piece cost him?

38. At what rate per annum will $3600 give $270 interest in one year 8 months?

Let r = the rate % per annum.

$$3600 \times \frac{r}{100} \times \frac{5}{3} = 270$$

39. What sum must be invested at 5% to give a quarterly income of $105?

40. What sum put at interest at 5% per annum will amount to $6000 in 1 year 9 months?

41. A father is 42 years old, and his son is $\frac{3}{7}$ as old. If both live, in how many years will the son be $\frac{3}{5}$ as old as his father?

42. Separate the number 145 into two parts so that the excess of the greater over 50 shall be 4 times the excess of 50 over the smaller.

43. If $\frac{2}{5}$ of a certain principal is invested at 5% and the remainder at 4%, the annual income from both investments is $660. Find the whole sum invested.

44. The width of a room is $\frac{2}{3}$ of its length. If the length were 4 feet less and the width 4 feet more, the room would be square. Find the dimensions of the room.

45. A man invested \$20,000, part of it at 5% and the remainder at 6%. The interest on the former for 2 years is the same as the interest on the latter for 2 years 6 months. How much was invested at each rate?

46. A man invested $\frac{1}{4}$ of his money in 4% bonds, $\frac{2}{5}$ of it in 5% bonds, and the remainder in 6% bonds, buying them all at par. His annual income from the whole investment amounts to \$2550. Find his whole investment.

GENERAL PROBLEMS

258. **A general problem** is a problem all of the numbers in which are general numbers.

It is therefore evident that the solution of a general problem involves a literal equation. For example:

The sum of two numbers is m, and the larger number is n times the smaller. Find the numbers.

Let x = the smaller number,
and nx = the larger number.

$$x + nx = m$$

Solving, $x = \dfrac{m}{1+n}$ and $nx = \dfrac{mn}{1+n}$

The result obtained in solving a general problem is a formula for solving all problems *of that type.*

To find the smaller number, divide the sum of the numbers by 1 *plus the ratio of the two numbers.*

To find the larger number, divide the product of the sum and ratio by 1 *plus the ratio of the numbers.*

These are the **rules** for finding *any two* numbers when their *sum* and their *ratio* are known.

259. **Generalization** in algebra is the process of solving a general problem and interpreting the formula obtained as a rule for solving all problems of that type.

Exercise 116

1. The larger of two numbers is 7 times the smaller, and their sum is 1488. Find the numbers.

$$\frac{m}{1+n}=1488\div 8 \qquad \frac{mn}{1+n}=(1488\times 7)\div 8$$

2. The smaller of two numbers is $\frac{3}{4}$ of the larger, and their sum is 21. Find the numbers.

3. If two numbers are added, the result is 2769, and one is $8\frac{3}{4}$ times the other. Find the two numbers.

4. The sum of two numbers is s, and the difference of the same numbers is d. Find the numbers.

Let x = the larger number,
and $s-x$ = the smaller number.

$$x-(s-x)=d$$

Solving, $x=\dfrac{s+d}{2}$ and $s-x=\dfrac{s-d}{2}$

5. Read these formulas as rules for finding two numbers when their sum and their difference are known.

6. The sum of two numbers is 768, and their difference is 116. Find the numbers.

$$\frac{s+d}{2}=\frac{768+116}{2} \text{ and } \frac{s-d}{2}=\frac{768-116}{2}$$

7. A man sold a piece of land for \$6800 and gained the same sum he would have lost, if he had sold it for \$5200. How much did he pay for the land?

8. The sum of two numbers is a, and m times the smaller is equal to n times the larger. Find the numbers.

Let L = the larger number,
and $a-L$ = the smaller number.

$$am-mL=nL$$

Solving, $L=\dfrac{am}{m+n}$ and $a-L=\dfrac{an}{m+n}$

SOLUTION OF FORMULAS

260. The student of physics and higher mathematics will often find it necessary to solve formulas. For example:

The distance passed over by any body moving with a uniform velocity in any number of units of time is the product of the velocity and the time.

This law expressed in a formula is—

$$\mathbf{d = vt}$$

Solving this equation for v and t, we have

$$v = d \div t \text{ and } t = d \div v$$

What is the average velocity of a train, if it runs 448 miles in 16 hours?

$$d \div t = 448 \div 16$$

261. The interest is the product of the principal, the rate expressed as hundredths, and the time.

$$\mathbf{i = prt}$$

It must be remembered that r in this formula represents the *rate per annum* and t the *number of years.*

Solving this formula, or literal equation, for p, r, and t, we have the following formulas:

$$p = i \div rt \qquad r = i \div pt \qquad t = i \div pr$$

1. What sum put at interest at 6% for 1 year 4 months will yield \$60 in interest?

$$i \div rt = 60 \div \tfrac{6}{100} \cdot \tfrac{4}{3} = 750$$

2. At what rate per annum will \$1300 amount to \$1391 in 1 year 4 months 24 days?

$$i \div pt = 91 \div 1300 \cdot \tfrac{504}{360} = .05$$

3. In how many years, months, and days will \$2200 amount to \$2345 at 5% per annum?

$$i \div pr = 143 \div 2200 \cdot \tfrac{5}{100} = \tfrac{143}{110} = 1 \text{ yr. } 3 \text{ mo. } 18 \text{ da.}$$

262. The ratio of the circumference of any circle to its diameter is approximately 3.1416. The exact value is represented by π.

The formulas for the circumference of a circle are

$$\mathbf{c=\pi d} \text{ and } \mathbf{c=2\pi r},$$

in which c is the circumference, d the diameter, and r the radius.

Solve $c=\pi d$ for d, and $c=2\pi r$ for r, and read results as rules for finding d and r.

263. Denoting the area by A, the base by b, and the altitude by h, the formulas for the area of a triangle are:

$$\mathbf{A=b\cdot\frac{h}{2}} \qquad \mathbf{A=h\cdot\frac{b}{2}} \qquad \mathbf{A=bh\cdot\frac{1}{2}}$$

The area of any triangle is the product of the base and half the altitude, the altitude and half the base, or half the product of the base and altitude.

Solve each of the above formulas for b and h, and read the results as rules for finding those dimensions.

264. Primes and Subscripts. Different but related numbers in a formula are often denoted by the same letter with different *primes* or *subscripts*.

Primes are accent marks written at the right and above the number; subscripts are small figures written at the right and below the number. For example, a', a'', a''', n_0, n_1, n_2, n_3.

These are read *a prime, a second, a third, n sub zero, n sub one, n sub two, n sub three*, respectively.

In the formula for the area of a trapezoid we shall find the two parallel bases denoted by b_1 and b_2.

265. Denoting the area by A, the two parallel sides or bases by b_1 and b_2, and the altitude by h, the formula for the area of any trapezoid is:

$$\mathbf{A=\frac{b_1+b_2}{2}\cdot h}$$

The area of a trapezoid is the product of half the sum of the two bases and the altitude.

Solve the above formula for b_1, b_2, and h, and read the results as rules for finding those dimensions.

Exercise 117 — General Formulas

1. Solve the formula $s=\dfrac{lr-a}{r-1}$ for a, r, and l.

2. A man sold a piece of land for n dollars and gained a per cent. How much did he pay for it?

3. Solve the formula $s=\dfrac{n(a+l)}{2}$ for a, l, and n.

4. What sum must be invested at $n\%$ per annum to yield a quarterly income of a dollars?

5. Solve $d_1w_1=d_2w_2$ for each general number.

6. By selling silk at m cents a yard, a merchant lost $b\%$. Find the cost per yard.

7. Solve $v_2t=v_1t+n$ for v_1, v_2, and t.

8. The length of a rectangular field is m times its width. Increasing its length a rods and its width b rods would increase its area n square rods. Find the dimensions.

9. What sum put at interest at r per cent per annum will amount to m dollars in n years?

10. Solve the formula $\dfrac{1}{q}-\dfrac{1}{p}=\dfrac{1}{f}$ for q, p, and f.

11. At what rate per annum will a dollars yield b dollars interest in c years?

12. Solve the formula $F=\dfrac{9C}{5}+32$ for C.

13. In how many years will the interest on a dollars amount to m dollars at $r\%$ per annum?

14. Solve the formula $\dfrac{a}{g}=\dfrac{h-l}{h+l}$ for a, g, h, and l.

5. $\begin{cases} 8x+5y=28 \\ 2y+4x=17 \end{cases}$

6. $\begin{cases} 6x+7y=-70 \\ 2y+2x=-25 \end{cases}$

7. $\begin{cases} 3x+2y=27 \\ 2x+3y=28 \end{cases}$

8. $\begin{cases} 3x-7y=-40 \\ 4x-5y=-23 \end{cases}$

9. $\begin{cases} 8x+5y=45 \\ 4x+3y=25 \end{cases}$

10. $\begin{cases} 2y-3x=-25 \\ 2x+5y=-40 \end{cases}$

11. $\begin{cases} 2x+3y=53 \\ 7x-2y=23 \end{cases}$

12. $\begin{cases} 5y-4x=-19 \\ 3x+2y=-26 \end{cases}$

13. $\begin{cases} 6y+6x=47 \\ 8x-3y=26 \end{cases}$

14. $\begin{cases} 6y-6x=-61 \\ 4x+9y=-20 \end{cases}$

15. $\begin{cases} 7y-2x=34 \\ 7x-2y=16 \end{cases}$

16. $\begin{cases} 4y-9x=-52 \\ 6x+8y=-16 \end{cases}$

17. $\begin{cases} 3y+6x=50 \\ 9x-4y=24 \end{cases}$

18. $\begin{cases} 3y-7x=-18 \\ 4x-7y=-32 \end{cases}$

267. In eliminating one unknown number from a system of fractional equations, it is often best to proceed without clearing the equations of fractions.

$$\frac{2x}{5}+\frac{5y}{3}=36 \qquad (1)$$

$$\frac{3x}{5}+\frac{3y}{4}=33 \qquad (2)$$

Multiplying (1) by 3 and (2) by 2 and subtracting the second result from the first, we have

$$\frac{15y}{3}-\frac{6y}{4}=42$$

From this equation the value of y is found to be 12. Substituting this value in (1), the value of x is 40.

268. Systems of fractional equations having the unknown numbers in the denominators, though not simple equations, may be solved as such for some of their roots.

In solving such equations, one of the unknown numbers should be eliminated without clearing of fractions. Thus,

$$\frac{2}{x}+\frac{4}{y}=22 \qquad (1)$$

$$\frac{3}{x}+\frac{5}{y}=30 \qquad (2)$$

Multiplying (1) by 3 and (2) by 2 and subtracting the second result from the first, we have

$$\frac{2}{y}=6$$

from which $y=\frac{1}{3}$; and substituting in (1),

$$x=\frac{1}{5}$$

Check by substituting in (1) and (2)

Exercise 119

Solve the following and check the first six:

1. $\begin{cases}\frac{1}{x}+\frac{1}{y}=25\\ \frac{1}{x}-\frac{1}{y}=15\end{cases}$

2. $\begin{cases}\frac{2}{x}+\frac{5}{y}=27\\ \frac{8}{x}-\frac{4}{y}=36\end{cases}$

3. $\begin{cases}\frac{1}{x}+\frac{1}{y}=2a\\ \frac{1}{x}-\frac{1}{y}=2b\end{cases}$

4. $\begin{cases}\frac{2}{x}+\frac{3}{y}=28\\ \frac{5}{x}-\frac{4}{y}=24\end{cases}$

5. $\begin{cases}\frac{a}{x}+\frac{b}{y}=4c\\ \frac{a}{x}-\frac{b}{y}=4d\end{cases}$

6. $\begin{cases}\frac{4}{x}-\frac{2}{y}=22\\ \frac{6}{y}-\frac{3}{x}=15\end{cases}$

In the following systems, first multiply each equation through by the l. c. m. of the known factors in the denominators:

7. $\begin{cases} \dfrac{2}{11x}-\dfrac{1}{11y}=1 \\ \dfrac{1}{5y}-\dfrac{1}{10x}=\dfrac{1}{2} \end{cases}$

8. $\begin{cases} \dfrac{1}{7x}+\dfrac{3}{14y}=2 \\ \dfrac{5}{8x}-\dfrac{1}{2y}=3 \end{cases}$

9. $\begin{cases} \dfrac{1}{5x}+\dfrac{1}{3y}=2 \\ \dfrac{1}{3x}+\dfrac{1}{y}=5\dfrac{1}{2} \end{cases}$

10. $\begin{cases} \dfrac{1}{3x}+\dfrac{2}{5y}=7 \\ \dfrac{1}{6x}+\dfrac{2}{5y}=6 \end{cases}$

Exercise 120

Solve and check, eliminating by any method (see §§ 119, 165, and 266):

1. $\begin{cases} 4y-2x=16 \\ 5x-3y=44 \end{cases}$

2. $\begin{cases} 5x+4y=-25 \\ 6y+7x=-45 \end{cases}$

3. $\begin{cases} 6x+5y=27 \\ 5x+6y=28 \end{cases}$

4. $\begin{cases} \frac{5}{6}x+\frac{2}{3}y=-20 \\ 9x+3y=-27 \end{cases}$

5. $\begin{cases} 2x-\frac{7}{9}y=36 \\ \frac{4}{5}x+2y=56 \end{cases}$

6. $\begin{cases} \frac{2}{3}x+4y=-44 \\ 3y+\frac{7}{8}x=-24 \end{cases}$

7. $\begin{cases} \dfrac{5x}{3}-\dfrac{2y}{5}=14 \\ \dfrac{2x}{3}+\dfrac{5y}{3}=33 \end{cases}$

8. $\begin{cases} \dfrac{3x}{4}+\dfrac{5y}{2}=-30 \\ \dfrac{4y}{3}+3x=-48 \end{cases}$

9. $\begin{cases} \dfrac{2a}{x}+\dfrac{2b}{y}=\dfrac{4d}{c} \\ \dfrac{2a}{x}-\dfrac{2b}{y}=\dfrac{4c}{d} \end{cases}$

10. $\begin{cases} \dfrac{5}{2x}+\dfrac{4}{3y}=-27 \\ \dfrac{5}{3y}+\dfrac{8}{3x}=-31 \end{cases}$

11. $\begin{cases} 5y+6x=47 \\ 4x+3y=35 \end{cases}$

12. $\begin{cases} \frac{2}{3}x-\frac{1}{5}y=-34 \\ 6y-2x=-30 \end{cases}$

33. $\begin{cases} ay+bx=2ab \\ by+ax=a^2+b^2 \end{cases}$

34. $\begin{cases} a+b=x+y \\ ax+a^2=by+b^2 \end{cases}$

35. $\begin{cases} y(a+n)=x(a-n)+an \\ x(a+n)=y(a-n)+3an \end{cases}$

36. $\begin{cases} ay+bx=a^2+b^2 \\ ax-by=-a^2-b^2 \end{cases}$

37. $\begin{cases} a^2-b^2=ax-by \\ (a+b)^2=ax+by \end{cases}$

38. $\begin{cases} \dfrac{1}{x_1}+\dfrac{1}{x_2}=35 \\ \dfrac{1}{x_1}-\dfrac{1}{x_2}=25 \end{cases}$

39. $\begin{cases} \dfrac{3x+4y+6}{2x-3y+1}=20 \\ \dfrac{4x+5y-2}{3x-3y-8}=17 \end{cases}$

40. $\begin{cases} \dfrac{2}{n_1}+\dfrac{3}{n_2}=29 \\ \dfrac{3}{n_1}+\dfrac{2}{n_2}=31 \end{cases}$

41. $\begin{cases} \dfrac{a+3}{2}+\dfrac{n-5}{6}=29 \\ \dfrac{n+7}{3}+\dfrac{a+9}{9}=20 \end{cases}$

42. $\begin{cases} \dfrac{a}{y_1}+\dfrac{a}{y_2}=12 \\ \dfrac{b}{y_1}-\dfrac{b}{y_2}=10 \end{cases}$

43. $\begin{cases} \dfrac{1}{p+1}+\dfrac{1}{q+1}=18 \\ \dfrac{3}{p+1}+\dfrac{2}{q+1}=45 \end{cases}$

44. $\begin{cases} \dfrac{1}{2x}+\dfrac{2}{3y}=18 \\ \dfrac{3}{4x}+\dfrac{4}{6y}=21 \end{cases}$

45. $\begin{cases} \dfrac{8}{n+5}+\dfrac{7}{s-4}=15 \\ \dfrac{12}{n+5}-\dfrac{2}{s-4}=10 \end{cases}$

46. $\begin{cases} \dfrac{x+y-1}{x-y+1}=10 \\ \dfrac{y-x+1}{x-y+1}=10 \end{cases}$

47. $\begin{cases} \dfrac{6}{x-1}-\dfrac{3}{y-1}=39 \\ \dfrac{5}{x-1}-\dfrac{1}{y-1}=37 \end{cases}$

48. $\begin{cases} \dfrac{1}{.3x}+\dfrac{2}{.6y}=100 \\ \dfrac{1}{.4x}+\dfrac{4}{.9y}=110 \end{cases}$

49. $\begin{cases} \dfrac{y}{a-d}+\dfrac{x}{a-c}=20 \\ \dfrac{y}{b-d}+\dfrac{x}{b-c}=20 \end{cases}$

50. $\begin{cases} \dfrac{3}{.02x} - \dfrac{3}{.04y} = 180 \\ \dfrac{3}{.04x} + \dfrac{5}{.08y} = 170 \end{cases}$

51. $\begin{cases} \dfrac{y}{n-s} + \dfrac{x}{n+s} = \dfrac{1}{n-s} \\ \dfrac{x}{n+s} - \dfrac{y}{n-s} = \dfrac{1}{n+s} \end{cases}$

PROBLEMS IN SIMULTANEOUS SIMPLE EQUATIONS

269. Many problems, which really contain two or more unknown numbers, are easily solved by the use of a single equation containing but one unknown number.

This method is advisable only when the relations between the unknown numbers are so simple that all of them can be expressed in terms of a single unknown.

In other problems it is better to introduce as many equations as there are unknown numbers.

When using a system of two or more equations to solve problems, enough conditions must be expressed in the problem to furnish as many independent equations as there are unknown numbers to be found.

Exercise 121 — Problems in Two Unknowns

Solve the following problems using equations involving two or more unknown numbers:

1. The sum of two numbers is 148, and their difference is 38. Find the numbers.

2. The larger of two numbers is $3\frac{1}{2}$ times the smaller, and their sum is 324. Find the numbers.

3. A man changed $7 into dimes and nickels, receiving 111 coins. How many of each did he have?

4. Of two consecutive numbers, $\frac{2}{5}$ of the smaller number exceeds $\frac{1}{3}$ of the larger by 6. Find the numbers.

5. Divide 118 into two parts so that 7 times the smaller part shall exceed 3 times the larger by 100.

6. If the pupils of a class are seated 3 on each bench, 5 pupils must stand. If 4 are put on each bench, one seat is not occupied. How many pupils are in the class?

7. Half the sum of two numbers is 73, and 4 times their difference is 128. Find the numbers.

8. The length of a rectangle exceeds its width by 14, and its perimeter is 116. Find the dimensions.

9. Find three numbers whose sum is 50, the first being 20 greater and the second 15 greater than the third.

10. In the equation $ax+by=32$, find a and b, if, when $x=4$, $y=2$; and if, when $x=10$, $y=-3$.

11. There are 4 more spokes in each front than in each rear wheel of a wagon, and in the 4 wheels there are 112 spokes. How many spokes are in each wheel?

12. The sum of three numbers is 59. The second is 8 greater then the first, and the third is 7 greater than the second. Find the numbers.

13. If 3 carpenters and 7 masons together receive a daily wage of \$61.20 and a mason receives 20 cents a day more than a carpenter, what is the daily wage of each?

14. Three tons of hard coal and 2 tons of soft coal cost \$32. At the same prices, 2 tons of hard coal and 6 tons of soft cost \$43.50. Find the price per ton of each.

15. The first of three numbers is twice the third, the second is 5 less than the first, and the sum of the three numbers is 55. Find the numbers.

16. A man invests part of \$3200 at 6% and the rest at 5%. If the annual income from the two amounts is \$180, what is the amount of each investment?

17. One dimension of a rectangle is 8, and one dimension

plus $\frac{2}{5}$ of the second plus $\frac{1}{2}$ of the third is 92; and if when 21 is added to the first, the sum is twice the third.

Let f = the first number,
and s = the second number,
and t = the third number.

$$f+s+t=168$$

$$\frac{2f}{3}+\frac{2s}{5}+\frac{t}{2}=92$$

$$f+21=2t$$

25. The average weight of 3 persons is 164 lb. The average weight of the first and second is 159 lb., and of the second and third 165 lb. Find the weight of each.

26. In a company of 29 persons there were 15 more adults than children and 4 more men than women. How many persons of each kind were there in the company?

27. How many bushels each of new wheat at \$1.05 a bushel and old wheat at 85¢ a bushel may be mixed to make a mixture of 200 bushels worth 90¢ a bushel?

28. The numerator of the larger of two fractions is 8, and the numerator of the smaller fraction is 5. The sum of the fractions is $1\frac{13}{18}$; and if the numerators are interchanged, their sum is $1\frac{8}{9}$. Find the fractions.

29. The sum of the three angles of a triangle is 180°. The sum of twice the first and the second exceeds the third by 90°; and the sum of the first and twice the third exceeds twice the second by 70°. Find the three angles of the triangle.

30. If a rectangle of paper were 4 in. shorter and 3 in. wider, the area would be 2 sq. in. less than it is. If a strip 2 in. wide is cut off on all sides, the area is diminished 184 sq. in. Find the dimensions.

31. In the equation $ax-by=20$, find x and y if when $a=7$, $b=5$; and if when $a=8$, $b=3\frac{1}{3}$.

32. The sum of the three digits of a number is 15. The digit in tens' place is half the sum of the other two; and if 198 is subtracted from the number, the first and last digits are interchanged. Find the number.

33. A man has \$49 in dollar bills, half-dollars, and quarters. Half of the dollars and $\frac{1}{5}$ of the half-dollars are worth \$15.50; $\frac{1}{7}$ of the half-dollars and $\frac{1}{3}$ of the quarters are worth \$5. How many coins has he?

34. A and B are 8 miles apart. If they set out at the same time and travel in the same direction, A will overtake B in 4 hours. If they travel toward each other, they will meet in $1\frac{1}{3}$ hours. At what rate does each travel?

35. A man bought a piece of land. At \$5 less per acre, he could have bought 40 acres more for the money; at \$4 more per acre, he could have bought 20 acres less for the money. Find the number of acres bought and the price per acre.

36. One woman paid \$2.75 for 7 lb. of coffee and 5 lb. of sugar; another paid \$2.05 for 3 lb. of coffee and 10 lb. of rice; another paid \$1.02 for 7 lb. of sugar and 6 lb. of rice. Find the uniform price of each per pound.

37. A harvest hand engaged to work two months, July and August, for his board and \$2.50 for each work-day. For each week-day he did not work he forfeited 50¢ for his board. The term of service contained 8 Sundays. At settlement he received \$123. How many days did he work?

38. The sums of the three pairs of sides of a triangle are 14, 15, and 17. How long is each side?

39. A classroom has 36 desks, some single and some double. The seating capacity of the room is 42. How many desks of each kind are there?

40. A sold 35 sheep to B and 25 to C. They each then had the same number. Before A made these sales, he had 10 more than B and C together. How many did each have at first?

41. A boy bought some peaches at the rate of 2 for 5¢ and some others at 3 for 5¢, paying \$6 for all of them. He sold them all at 40¢ a dozen and made a profit of \$4. How many did he buy at each price?

42. A has his money invested at 4%, B at 5%, and C at 6%. A's and B's annual interest together is \$398; B's and C's together is \$441.50; and A's and C's together is \$409.50. How much money has each one invested?

43. The width of a rectangular sheet of paper is 6 inches greater than half its length. If a strip 3 inches wide were cut off on the four sides, it would contain 360 square inches. Find the dimensions of the paper.

44. If the sum of $\frac{2}{3}$ of the first of three numbers and $\frac{3}{4}$ of the second is 118, the sum of $\frac{2}{3}$ of the second and $\frac{3}{4}$ of the third is 93, and the sum of $\frac{2}{3}$ of the third and $\frac{3}{4}$ of the first is 112, what are the numbers?

45. A street car has 12 short and 4 long seats. When the seats are all occupied, 56 persons are seated, each long seat holding 6 more passengers than each short one. How many passengers does each kind of seat accommodate?

46. The sum of two sides of a triangle is 58 feet, and the difference is 14 feet. The perimeter of the triangle is 103 feet. Find the length of each side.

47. In 8 months a sum of money at simple interest amounts to \$780. At the same rate, in 14 months it amounts to \$802.50. Find the sum invested and the rate.

THREE OR MORE UNKNOWN NUMBERS

270. To find the values of *three* unknown numbers, *three* independent simultaneous equations are necessary. In general there must be as many independent simultaneous equations as there are unknown numbers to be found.

In solving systems of several simultaneous equations, it is best to eliminate by addition or subtraction. Thus,

To solve $\begin{cases} 6x+4y+2z=32 & (1) \\ 2x+3y+3z=25 & (2) \\ 4x-2y+5z=22 & (3) \end{cases}$

Multiplying (2) by 3, $6x+9y+9z=75$ (4)

$\underline{6x+4y+2z=32}$ (1)

Subtracting (1) from (4), $5y+7z=43$ (5)

Multiplying (2) by 2, $4x+6y+6z=50$ (6)

$\underline{4x-2y+5z=22}$ (3)

Subtracting (3) from (6), $8y+z=28$ (7)

Multiplying (7) by 7, $56y+7z=196$ (8)

$\underline{5y+7z=\ \ 43}$ (5)

Subtracting (5) from (8), $51y=153$ (9)

Dividing (9) by 51, $y=3$

Substituting the value of y in (5) or in (7), the value of z is found to be 4. Substituting the values of y and z in (1), (2), or (3), the value of x is found to be 2.

Check by substituting the calculated values of x, y, and z in (1), (2), and (3).

Solve, $\begin{cases} 6x-3y-2z=15 & (1) \\ 5x+2y-9z=13 & (2) \\ 4x+3z=33 & (3) \end{cases}$

Combine (1) and (2) and eliminate y. Then combine the new equation found with (3) and eliminate either x or z.

271. One or more of a system of equations may *not* contain all the unknown numbers.

Solve:
$$\begin{cases} 3x+3y=33 & (1) \\ 6y-3z=15 & (2) \\ 6z-2x=32 & (3) \end{cases}$$

Multiplying (1) by 2, $$6x+6y=66 \quad (4)$$
$$-3z+6y=15 \quad (2)$$

Subtracting (2) from (4), $$6x+3z=51 \quad (5)$$

Multiplying (3) by 3, $$-6x+18z=96 \quad (6)$$

Adding (5) and (6), $$21z=147$$
$$z=7$$

Substituting the value of z in (2) and solving,
$$y=6$$

Substituting the value of y in (1) and solving,
$$x=5$$

Check by substituting $x=5$, $y=6$, and $z=7$ in (1), (2) and (3)

Exercise 122

Solve the following systems of equations:

1. $\begin{cases} x+y=18 \\ y+z=19 \\ x+z=17 \end{cases}$

2. $\begin{cases} 2x+3y-4z=16 \\ 4x-2y+3z=45 \\ 8x-3y-4z=28 \end{cases}$

3. $\begin{cases} m-n=13 \\ n-p=14 \\ m-p=27 \end{cases}$

4. $\begin{cases} 2x+3y-4z=12 \\ 3x-3y+2z=30 \\ 4x-6y+5z=45 \end{cases}$

5. $\begin{cases} 4x-2z=18 \\ 3x-2y=17 \\ 7y-3z=26 \end{cases}$

6. $\begin{cases} 3x+5y+28=3z \\ 2x+4y+24=2z \\ 4x+2y-14=5z \end{cases}$

7. $\begin{cases} 2x+3y=-32 \\ 6y-2z=-26 \\ 4x+3z=-43 \end{cases}$

8. $\begin{cases} 3x-15+2z=4y \\ 3y+2x-15=3z \\ 5y-22+4z=5x \end{cases}$

9. $\begin{cases} 2x+4y+3z=35 \\ x+2z+3y=23 \\ 6z-5y+3x=27 \end{cases}$

10. $\begin{cases} 3x+3z-4y=12 \\ 3y-4z+4x=10 \\ 3y+2x-2z=14 \end{cases}$

11. $\begin{cases} 4y-3z+2x=22 \\ 3x+3y-2z=22 \\ 5z+4x-2y=18 \end{cases}$

12. $\begin{cases} 2x+3z-3y=14 \\ 3y+z+2x=16 \\ 5x+4y-2z=15 \end{cases}$

13. $\begin{cases} 4y+5z+3x=38 \\ 3y+5x+4z=35 \\ 4x+3z+5y=35 \end{cases}$

14. $\begin{cases} 3x+4y+2z=29 \\ 4y+3z+2x=27 \\ 4x+3z+2y=33 \end{cases}$

15. $\begin{cases} x-y=4 \\ x-z=7 \\ y-z=3 \end{cases}$

16. $\begin{cases} x+y=1 \\ x-z=1 \\ y-z=1 \end{cases}$

17. $\begin{cases} x+y=a \\ x+z=b \\ y+z=c \end{cases}$

18. $\begin{cases} \frac{1}{x}+\frac{1}{y}=9 \\ \frac{1}{y}+\frac{1}{z}=8 \\ \frac{1}{z}+\frac{1}{x}=7 \end{cases}$

19. $\begin{cases} \frac{x}{a}+\frac{y}{b}=4 \\ \frac{y}{b}+\frac{z}{c}=4 \\ \frac{z}{c}+\frac{x}{a}=4 \end{cases}$

20. $\begin{cases} \frac{x}{5}+\frac{y}{7}=7 \\ \frac{y}{4}-\frac{z}{4}=4 \\ \frac{z}{3}+\frac{x}{3}=9 \end{cases}$

21. $\begin{cases} \frac{1}{x}-\frac{1}{y}=2 \\ \frac{1}{y}-\frac{1}{z}=5 \\ \frac{1}{x}-\frac{1}{z}=7 \end{cases}$

22. $\begin{cases} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{x}{a}+\frac{z}{c}=1 \\ \frac{y}{b}+\frac{x}{a}=1 \end{cases}$

23. $\begin{cases} \frac{1}{x}+\frac{1}{y}=n \\ \frac{1}{y}+\frac{1}{z}=p \\ \frac{1}{z}+\frac{1}{x}=q \end{cases}$

9. $\dfrac{1}{\frac{1}{2}}$ 10. $\dfrac{8}{8b}$ 11. $\dfrac{\frac{2}{3}}{\frac{4}{5}}$ 12. $\dfrac{\frac{8}{9}}{\frac{6}{7}}$

13. $\dfrac{10}{2\frac{1}{2}}$ 14. $\dfrac{a}{a(x+3)}$ 15. $\dfrac{a}{b} : 1$ 16. $m^2-n^2 : (m-n)^2$

273. Measuring is Ratioing. To measure any kind of magnitude is to find its ratio to some standard unit of the kind of magnitude being measured. Measured magnitudes are expressed by so-called concrete numbers, such as 6 in., 10 ft., 4 lb., 20 acres, 5 days, etc.

The value of a ratio of two such numbers is calculated by first *expressing both numbers in a common unit,* and then finding the value of the ratio of these equivalents. Thus, the ratio of 12 in. to 3 ft. is not $\frac{12}{3}$, but 12 in. to 36 in., or $\frac{12}{36}$, or $\frac{1}{3}$.

If the two magnitudes cannot be expressed in a common unit, it is without meaning to speak of their ratios.

Exercise 123

Simplify the following ratios:

1. $18 : 12$ 2. $24y : 8y$ 3. $2\frac{4}{5} : 1\frac{2}{5}$

4. 1 mi. : 660 ft. 5. 7 da. : 7 hr. 6. 16 lb. : 4 oz.

7. 100 lb. : 1 ton 8. $\dfrac{a(x+y)}{x+y}$ 9. $\dfrac{a(x+y)}{a}$

10. $\dfrac{3(x-y)}{x-y}$ 11. $\dfrac{a^2-b^2}{(a+b)^2}$ 12. $\dfrac{a^2-b^2}{(a-b)^2}$

13. $\dfrac{a^3-b^3}{a^2+ab+b^2}$ 14. $\dfrac{a^3+b^3}{a^2-ab+b^2}$ 15. $\dfrac{x^3-y^3}{(x-y)^3}$

16. $\dfrac{\left(3-\frac{1}{n}\right)}{\left(3+\frac{1}{n}\right)}$ 17. $\dfrac{x-5}{x^2-3x-10}$ 18. $\dfrac{a^2+b^2}{a^4-b^4}$

274. A **ratio of greater inequality** is a ratio in which the antecedent is greater than the consequent. Thus, 7 : 5 is a ratio of greater inequality.

275. A **ratio of less inequality** is a ratio in which the antecedent is less than the consequent. Thus, 5 : 7 is a ratio of less inequality.

276. Theorem. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same positive number to both its terms.*

Illustration. (*a*) For a ratio of greater inequality.

Let the ratio be 5 : 3, or $\frac{5}{3}$, and the positive number, n, be added to both terms giving the ratio $5+n : 3+n$, or $\frac{5+n}{3+n}$.

Then dividing:

$$\begin{array}{r|l} 3+n) & 5+\ \ n\ (\tfrac{5}{3} \\ & \underline{5+\tfrac{5}{3}n} \\ & \ \ -\tfrac{2}{3}n \end{array}$$

Or, since *dividend = divisor × quotient + remainder*,

$$\frac{5+n}{3+n}=\frac{5}{3}-\frac{2n}{3(3+n)}, \text{ or } \frac{5+n}{3+n} \text{ is less than } \frac{5}{3}.$$

(*b*) An illustration for a ratio of less inequality is left to the student.

Exercise 124 — Problems

1. Divide a scantling 16 ft. long into two parts that are to each other as 3 : 5.

Call one part $3x$ and the other $5x$. Note that $3x+5x=16$, and find x, $3x$, and $5x$.

2. Divide the number 80 into two parts that are to each other as 2 : 3.

3. Divide an 18-foot scantling into two parts that are as 3 : 5.

4. Separate 121 into two parts that are to each other as 3 : 8.

5. The value of a fraction is $\frac{2}{3}$. If both terms are increased by 2, the value of the resulting fraction is $\frac{5}{7}$. Find the original fraction.

Notice that the numerator of the original fraction is to the denominator as 2 : 3.

6. The numerator is to the denominator of a fraction as 3 : 4. If the numerator is increased and the denominator diminished by 5, the value of the resulting fraction is $\frac{4}{3}$. Find the original fraction.

7. The value of a fraction is $\frac{9}{10}$. If 4 is subtracted from both terms the resulting fraction has the value $\frac{7}{8}$. Find the original fraction.

8. The ratio of the areas of two fields is $\frac{2}{5}$. The larger field is 25 acres. Find the area of the smaller field.

9. The areas of fields of the same shape are as the squares of their corresponding sides. How do the areas of two fields compare if a pair of corresponding sides are as 7 : 13?

PROPORTION

277. A **proportion** is an equation of ratios.

Examples: $\frac{2}{3}=\frac{6}{9}$ and $5x : 4x = 5 : 4$.

Four numbers, as a, b, c, and d, are said to be *in proportion*, or *proportional*, if $a : b = c : d$, or $\frac{a}{b}=\frac{c}{d}$.

278. The terms of the ratios are called **terms** of the proportion.

279. Extremes and Means. The first and fourth terms of the proportion are the **extremes** and the second and third terms are the **means**.

7. $\frac{12}{36}=\frac{x}{3}$ 8. $\frac{x-2}{21}=\frac{1}{3}$ 9. $\frac{x+1}{x-1}=\frac{3}{1}$

10. $\frac{1}{x+a}=\frac{1}{3a}$

282. Mean Proportional. If the second and third terms of a proportion are the same number, as in $5:15=15:45$, this number, the 15, is a **mean proportional** between the extremes.

Thus, in $a:x=x:b$, x is a mean proportional between a and b, and we have:

$$x^2=ab, \text{ whence } x=\sqrt{ab}, \text{ or, in words:}$$

A mean proportional between two numbers is the square root of their product.

283. A Third Proportional. In the proportion $a:b=b:c$, the number c is a **third proportional** to a and b.

Thus in $\frac{5}{20}=\frac{20}{80}$, 80 is a third proportional to 5 and 20.

284. A Fourth Proportional. A fourth proportional to the three numbers, a, b, and c, is the number d in the proportion $\frac{a}{b}=\frac{c}{d}$. It is the number, which with the three given numbers, completes a four-termed proportion.

Thus in $\frac{7}{13}=\frac{21}{39}$, 39 is the fourth proportional to 7, 13, and 21.

Exercise 127

Find mean proportionals between:

1. 3 and 27
2. 4 and 16
3. 1 and 81
4. a and b
5. $\frac{1}{4}$ and $\frac{1}{25}$
6. 1 and x^2
7. $\frac{4}{a}$ and $\frac{25}{ax^2}$
8. $a+b$ and $9(a+b)^3$

Find third proportionals to the following:

9. 2 and 6

10. 4 and 9

11. 2 and 22

12. $\frac{1}{4}$ and 2

13. $\frac{2}{5}$ and -8

14. $x-y$ and $x+y$

15. $\frac{1}{a}$ and $\frac{a}{x}$

16. $x+y$ and x^2-y^2

Find fourth proportionals to the following:

17. 4, 8, and 12

18. 12, 3, and 1

19. 5, 6, and $12\frac{1}{2}$

20. a, x, and y

21. m, n, and p

22. a, a^2, and a^4

23. $m+n$, $m-n$, and m^2-n^2

24. x^4, x^7, and x^9

25. $x+1$, $\frac{1}{x-1}$, and $x-1$

PRINCIPLES OF PROPORTION

285. Since each of the following products is 24, we may write

$$2 \cdot 12 = 3 \cdot 8.$$

Using only the four numbers of these two products, we may write the two columns below, the first being proportions and the second, not proportions.

Test by § 281 the expressions of both columns and show that the expressions of the first column meet the test, while those in the second column do not.

PROPORTIONS	EXPRESSIONS NOT PROPORTIONS
1. $2:3=8:12$	**1.** $2:12=3:8$
2. $2:8=12:3$	**2.** $2:8=12:3$
3. $12:3=8:2$	**3.** $12:2=3:8$
4. $12:8=3:2$	**4.** $12:8=2:3$
5. $3:12=2:8$	**5.** $3:8=12:2$

6. $3 : 2 = 12 : 8$
7. $8 : 12 = 2 : 3$
8. $8 : 2 = 12 : 3$

6. $3 : 2 = 8 : 12$
7. $8 : 3 = 12 : 2$
8. $8 : 12 = 3 : 2$

Notice that in the first column the proportions are made by using *both* factors of one of the products as *means*, and both factors of the other product as *extremes*. In the second column notice that this plan is not observed, and that the expressions obtained are *not* proportions.

286. Principle. *If the product of two numbers equals the product of two other numbers, the factors of either product may be made the means and those of the other product the extremes of a proportion.*

Suppose $a \cdot d = b \cdot c$

To prove $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{c} = \frac{b}{d}$, etc.

Proof. Divide both members of $a \cdot d = b \cdot c$ by bd, and simplify, obtaining $\frac{a}{b} = \frac{c}{d}$.

Also divide both sides of $a \cdot d = b \cdot c$ by cd, and obtain $\frac{a}{c} = \frac{b}{d}$, etc. Other proportions are proved similarly.

See how many of the 8 possible proportions you can write from the equation $a \cdot d = b \cdot c$

You should be able to write *two*, beginning with any one of the 4 letters.

Exercise 128

Write all the proportions you can from the following equations:

1. $3 \cdot 12 = 4 \cdot 9$
2. $2 \cdot 25 = 5 \cdot 10$
3. $3 \cdot 7 = 21 \cdot 1$
4. $m \cdot q = n \cdot p$

287. Just as equations may be derived from other equations so may proportions be derived from other proportions. The principles for deriving proportions from proportions are now to be established.

If $\frac{a}{b}=\frac{c}{d}$, then (1) $\frac{a+b}{a}=\frac{c+d}{c}$, or (2) $\frac{a+b}{b}=\frac{c+d}{d}$.

To prove (1) and (2), proceed *by analysis*, thus:

ANALYSIS

Assume (1) $\frac{a+b}{a}=\frac{c+d}{c}$, or (2) $\frac{a+b}{b}=\frac{c+d}{d}$.

Reduce the improper fractions to mixed numbers thus:

$$1+\frac{b}{a}=1+\frac{d}{c} \quad \text{or} \quad \frac{a}{b}+1=\frac{c}{d}+1$$

Whence, $\frac{b}{a}=\frac{d}{c}$ or $\frac{a}{b}=\frac{c}{d}$

PROOF

We may now construct the proof, by reversing the steps just given.

We know that $\frac{b}{a}=\frac{d}{c}$ if $\frac{a}{b}=\frac{c}{d}$. Why?

Add 1 to both sides of the equations:

$$1+\frac{b}{a}=1+\frac{d}{c} \text{ and } \frac{a}{b}+1=\frac{c}{d}+1$$

Reducing to improper fractions we have:

$$\frac{a+b}{a}=\frac{c+d}{c} \text{ and } \frac{a+b}{b}=\frac{c+d}{d}.$$

When either of the last two proportions is inferred directly from $\frac{a}{b}=\frac{c}{d}$, the proportion, $\frac{a}{b}=\frac{c}{d}$, is said to be taken **by addition.**

Proportion *by addition* is often called proportion *by composition.*

291. Proportion by Subtraction. *If four numbers are in proportion, they will be in proportion* **by subtraction.** *That is, the difference of the terms of each ratio form a proportion with either the antecedents or the consequents of the ratios.*

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{a}=\frac{c-d}{c}$, or $\frac{a-b}{b}=\frac{c-d}{d}$

Use the method of analysis just as it was used above.

Proportion *by subtraction* is often called proportion *by division.*

292. Proportion by Addition and Subtraction. *If four numbers are in proportion they will be in proportion* ***by addition and subtraction.***

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Combine the results of the principles of §§ 290 and 291.

Proportion *by addition and subtraction* is often called proportion *by composition and division.*

Exercise 129

1. From each of the following, write a proportion (1) by alternation, (2) by inversion, (3) by addition, (4) by subtraction, and (5) by addition and subtraction:

1. $\frac{3}{2}=\frac{9}{6}$ 2. $\frac{2}{9}=\frac{14}{63}$ 3. $\frac{6}{-11}=\frac{-42}{77}$ 4. $\frac{8}{3}=\frac{m+n}{m-n}$

2. From each of the following equations write a proportion, commencing with each of the four factors; then take each proportion (1) by alternation, (2) by inversion, and (3) by addition and subtraction:

1. $2\cdot 9=3\cdot 6$ 2. $3\cdot 8=2\cdot 12$

3. $\frac{3}{4}\cdot 7=3\cdot\frac{7}{4}$ 4. $2.7\cdot 3=9\cdot 0.9$

3. Find values of x in the following proportions:

1. $\frac{x-7}{7}=\frac{6}{4}$ 2. $\frac{x+5}{x-3}=\frac{5}{1}$

3. $\frac{x-2}{x+2}=\frac{7}{11}$ 4. $\frac{3x+5}{5x-5}=\frac{11}{5}$

5. $\frac{x+4}{x-3}=\frac{1}{8}$

6. $\frac{x+6}{x-6}=-\frac{1}{5}$

4. Divide 91 into two parts that are to each other as $2\frac{1}{2} : 3\frac{1}{3}$.

5. Divide m into two parts that are to each other as $a : b$.

6. The difference between two numbers that are to each other as $a : b$, is d. Find them.

7. What number must be added to each term of $3 : 6 = 4 : 8$ to give another proportion?

8. By what number must each factor of the products $25 \cdot 51$ and $31 \cdot 40$ be reduced that the products may be equal?

9. By what number must each factor of the product $30 \cdot 147$ be reduced and each factor of $14 \cdot 62$ be increased, to make the products equal?

10. What number must be added to both m and n to give sums which are to each other as $a : b$?

11. What number added to m and subtracted from n gives numbers to each other as $a : b$?

12. The value of a fraction is $\frac{3}{4}$. Increasing numerator and denominator by 2 gives a fraction whose value is $\frac{4}{5}$. What is the fraction?

13. The denominator of a fraction is 6 greater than the numerator. Reducing both terms by 1 gives a fraction whose value is $\frac{1}{2}$. Find the fraction.

14. If the denominator of a fraction whose value is $\frac{3}{5}$, is increased and the numerator decreased by 3, the value of the resulting fraction is $\frac{1}{3}$. Find the fraction.

15. By what number must both terms of $\frac{29}{69}$ be increased to give a fraction whose value is $\frac{3}{10}$?

16. The value of a fraction is $\frac{4}{5}$. If 7 is added to the numerator and 2 to the denominator, the reciprocal value of the original fraction is obtained. Find the original fraction.

Exercise 130

1. Assume the amount, w, of water in a barrel to *vary as* the time, t, since the in-flow began. Write the general law for the amount of water in the barrel at time, t.

Ans. $$w = qt$$

2. Suppose that after 3 minutes of flow there are 36 qt. of water in the barrel. Find q and state the law definitely.

Substitute $w = 36$, and $t = 3$ in the general law, $w = qt$, obtaining $36 = q \cdot 3$, or $q = 12$, and the *definite form* of the law is then

$$w = 12t$$

3. After 2 minutes of flow how many quarts will have passed into the barrel?

Substitute $t = 2$ in $w = 12t$, obtaining $w = 12 \cdot 2 = 24$.

4. The amount of water in a cistern is assumed to vary as the square of the time, t, since the in-flow through a tube began. Express the general law connecting w and t.

General law: $$w = qt^2$$

5. Suppose that after 5 minutes there are 225 qt. in the cistern. Find q and state the law definitely.

In $w = qt^2$, put $w = 225$ and $t = 5$, obtaining:

$$225 = q \cdot 25, \text{ or } q = 9,$$

Definite law, $$w = 9t^2.$$

6. Find the quantity of water in the cistern after 3 minutes of flow.

In $w = 9t^2$, substitute $t = 3$, giving

$$w = 9 \cdot 9 = 81. \quad \text{(81 qt. in cistern)}$$

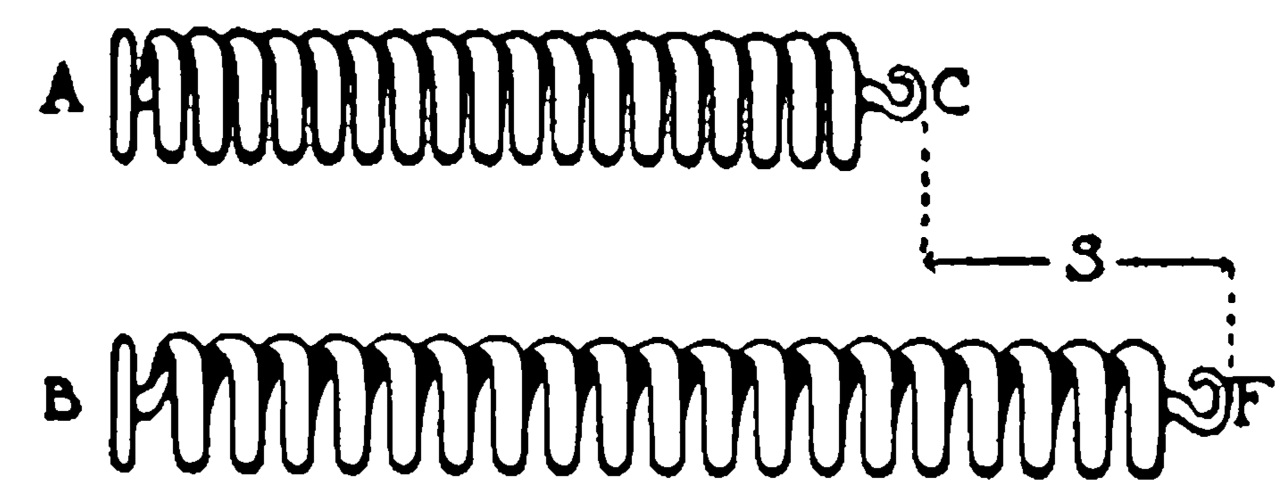
7. After how long will there be 900 qt. in the cistern?

$$900 = 9 \cdot t^2, \text{ or } t^2 = 100, \text{ and } t = 10 \text{ (after 10 min.)}$$

8. If $y \propto x$ and $y = 10$ when $x = 5$, what is the law connecting x and y?

We have, first, $y = kx$.
Making $y = 10$ and $x = 5$, $10 = 5k$
Therefore, $k = 2$
Hence, $y = 2x$.

9. When a spring is stretched by a force, F, the amount of stretch, s, varies as the strength of the force, F. Express the general law of stretch.



The stretch, s, varies as the force, F.

How is this law shown by the graduation marks of an ordinary spring balance?

10. When the force is 20 lb., the stretch is 5 inches. Find k, and express the law definitely.

11. How much would a force of 32 lb. stretch the spring?

12. How many pounds of force would have to be exerted to give a stretch of 10 inches?

13. The area, A, of a square varies as the square of a side, s. When $s = 5$, $A = 25$. Find k, and express the law connecting A and s in definite form. Have you met this law before?

14. If the altitude of a rectangle is constant, the area, A, of the rectangle varies as the base, x. Write the general law.

15. If the base is 12, the area is 96. Express the law in definite form.

16. The area, A, of an equilateral triangle varies as the square of a side, s. Express the law connecting A and s in general form.

17. When the side of the triangle is 6 the area is $3\sqrt{\frac{3}{2}}$. Find k, and express the law in definite form.

18. The work, w, of a machine varies as the number of hours, h, that it runs. Write the general law of work for the machine.

19. Working 3 hours, the machine does 59,400 foot-tons of work. Express the law of the machine in definite form.

20. How much work would the machine do in 1 minute, or $\frac{1}{60}$ of an hour?

CHAPTER XXI

POWERS. ROOTS

INVOLUTION

296. In §§ 140, 183, 185, and 187 we learned how to raise monomials to any power, also how to square binomials and polynomials. Those sections should be reviewed here.

297. Involution is the process of raising a number to a power whose exponent is a positive integer.

Involution is indicated by an exponent, and the exponent which indicates how many times the number is taken as a factor is called the **exponent of the power.** Thus,

$(2a^2)^3$ $\qquad$ $(a+x)^4$ $\qquad$ $(2a-3b)^3$

298. The **base** of a power in involution is the number which is raised to a power.

It has been shown that to multiply any power of a base by any power of the same base, the exponents are **added.** Thus,

$a^2 \times a = a^3$ $\qquad$ $a^2 \times a^3 = a^5$ $\qquad$ $a^4 \times a^2 = a^6$

The expression of this law in general numbers is

$$\mathbf{a}^m \times \mathbf{a}^n = \mathbf{a}^{m+n}.$$

299. It has been shown that to divide any power of a base by any lower power of the same base, the exponent of the divisor is **subtracted** from the exponent of the dividend. Thus,

$a^4 \div a^2 = a^2$ $\qquad$ $a^5 \div a^4 = a$ $\qquad$ $a^6 \div a^2 = a^4$

POWER OF A PRODUCT

302. Let a and b represent any two numbers and n any positive integer. Then $(ab)^n$ will represent any power of the product of any two numbers. By definition of a power:

$$(ab)^n = ab \cdot ab \cdot ab \cdot ab \cdot ab \cdots \text{ to } n \text{ factors,}$$
$$= (aaa \cdots \text{ to } n \text{ factors})(bbb \cdots \text{ to } n \text{ factors}),$$
$$= a^n b^n$$

The nth power of the product of two or more numbers is the product of the nth powers of the numbers.

The expression of this law in general numbers is

$$(\mathbf{ab})^n = \mathbf{a}^n\mathbf{b}^n.$$

In a similar manner it may be shown that the law holds for the product of any number of factors. Thus,

$$(2a^2b^nc)^3 = 2^3a^6b^{3n}c^3 = 8a^6b^{3n}c^3$$

Exercise 132

Write the power of each of the following:

1. $(2a^2)^3$	2. $(2^2 \cdot 3^2)^2$	3. $(a^nb^n)^2$	4. $(3ab^nc)^3$
5. $(3x^3)^4$	6. $(4^3 \cdot 5^4)^2$	7. $(x^2y^3)^n$	8. $(2ac^2x)^n$

POWER OF A FRACTION

303. We have seen that:

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \text{ to } n \text{ factors,}$$
$$= \frac{a \cdot a \cdot a \cdots \text{ to } n \text{ factors}}{b \cdot b \cdot b \cdots \text{ to } n \text{ factors}},$$
$$= \frac{a^n}{b^n}$$

The nth power of a fraction is the nth power of the numerator divided by the nth power of the denominator.

The expression of this law in general numbers is

$$\left(\frac{\mathbf{a}}{\mathbf{b}}\right)^n = \frac{\mathbf{a}^n}{\mathbf{b}^n}$$

Exercise 133

Give the power of each of the following:

1. $\left(\frac{1}{4a}\right)^2$ 2. $\left(-\frac{ab}{xy}\right)^3$ 3. $\left(\frac{a^2x^3}{b^3y^2}\right)^2$ 4. $\left(-\frac{a}{2b}\right)^3$

5. $\left(\frac{1}{3x}\right)^3$ 6. $\left(-\frac{4a}{5b}\right)^2$ 7. $\left(\frac{a^nb^n}{x^2y^2}\right)^3$ 8. $\left(-\frac{x}{3y}\right)^4$

9. $\left(\frac{1}{2n}\right)^4$ 10. $\left(-\frac{ax}{by}\right)^4$ 11. $\left(\frac{a^2c^2}{x^3y^3}\right)^n$ 12. $\left(-\frac{5}{2a}\right)^2$

13. $\left(\frac{2}{5a}\right)^2$ 14. $\left(-\frac{2a}{3b}\right)^3$ 15. $\left(\frac{a^2b^n}{x^3y^n}\right)^m$ 16. $\left(-\frac{3}{2x}\right)^3$

POWERS OF BINOMIALS

304. By multiplication, the following powers of $a+b$ and $a-b$ may be obtained:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

$$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$$

305. From an examination of these powers, or ***expansions***, considering n to represent the exponent of the power, the following laws hold in each expansion:

1. *Every term of the expansion, except the last, contains* a; *and every term, except the first, contains* b.

2. *The number of terms in the expansion is* $n+1$; *that is, it is* 1 *greater than the exponent of the power.*

3. *If both terms of the binomial are positive, all the terms of the expansion are positive.*

4. *If the second term is negative, the odd terms of the expansion are positive, the even terms negative.*

5. *The exponent of* **a** *in the first term of the expansion is* **n**, *and it diminishes by* 1 *in each succeeding term.*

6. *The exponent of* **b** *in the second term of the expansion is* 1, *and it increases by* 1 *in each succeeding term.*

7. *The coefficient of the first term of the expansion is* 1; *the coefficient of the second term is* **n**; *and the coefficient of any succeeding term is found by multiplying the coefficient of the preceding term by the exponent of* **a** *in that term, and dividing the product by a number* 1 *greater than the exponent of* **b** *in that term.*

The statement of these laws constitutes what is called the **binomial theorem.** The theorem is true of all the examples given. We shall take it for granted that it is true for *any positive integral power* of a binomial, but a general proof lies beyond the scope of this book.

Students will find it helpful to memorize the coefficients of the 1st, 2d, 3d, 4th, 5th, and 6th powers.

306. These coefficients may be arranged in a table forming what is known as **Pascal's Triangle,** as follows:

Coefficients of 1st power:	1	1					
Coefficients of 2d power:	1		1				
Coefficients of 3d power:	1		3	1			
Coefficients of 4th power:	1		6	4	1		
Coefficients of 5th power:	1		10	10	5	1	
Coefficients of 6th power:	1	4	15	20	15	6	1

Each coefficient is the *sum* of the number above it and the number to the left of the latter.

The coefficients of two terms equally distant from the first and last terms of the expansion are equal.

EVOLUTION

308. A root of a number is one of the *equal* factors whose product is the number.

Thus, 2 is a root of 8, 16, 32, 64, etc.

3 is a root of 9, 27, 81, 243, etc.

5 is a root of 25, 125, 625, etc

Roots are named from the number of equal factors that make the number. See two definitions § 190.

What root of 16 is 2? What root of 16 is 4? What root of 64 is 2? What root of 64 is 4? What root of 81 is 3?

309. Evolution is the process of finding a root, or one of the equal factors, of a number.

Evolution is indicated by the **radical sign** $\sqrt{\ }$ which is placed before the number.

The radical sign alone indicates the square root. If any other root is required, it is indicated by a small figure called the *index of the root*, written in the $\surd$ of the radical sign, thus:

$\sqrt{16}$, $\sqrt[3]{35}$, $\sqrt[4]{16}$, $\sqrt[5]{45}$, $\sqrt[6]{64}$

A symbol of aggregation with the radical sign indicates the part of the expression that is affected by the sign.

Thus, $\sqrt{25}+24$ means the sum of $\sqrt{25}$ and 24, while $\sqrt{25+24}$ means the square root of the sum of 25 and 24. The long bar above is a **vinculum**. See § 65.

Any root of a number indicated by the radical sign is called a **radical.**

Since evolution is the reverse of involution, the **nth** *root of* **a** *is a number the* **nth** *power of which is* **a.**

ROOT OF A POWER

310. Since $(a^m)^n = a^{mn}$,

$$\sqrt[n]{a^{mn}} = a^m,$$

by extracting the nth root of both members,

The ***n****th root of a power is obtained by dividing the exponent of the power by* ***n****.*

Exercise 137

1. How would you find the square root of a power? The cube root? The fourth root? The fifth root?

2. Give the indicated root of each of the following:

1. $\sqrt{a^6}$ **2.** $\sqrt[3]{x^9}$ **3.** $\sqrt[4]{b^8}$ **4.** $\sqrt{a^{2n}}$ **5.** $\sqrt[n]{x^{2n}}$

ROOT OF A PRODUCT

311. Since $(ab)^n = a^n b^n$, then

$$\sqrt[n]{a^n b^n} = ab. \quad \text{Why?}$$

The ***n****th root of the product of two or more factors is the product of the* ***n****th root of the factors.*

Exercise 138

Find the indicated root of each of the following:

1. $\sqrt{a^2x^4}$ **2.** $\sqrt[3]{27x^3}$ **3.** $\sqrt[4]{16a^8}$ **4.** $\sqrt[n]{a^n b^{2n}}$

5. $\sqrt[3]{x^6y^9}$ **6.** $\sqrt{49a^8}$ **7.** $\sqrt[4]{81x^4}$ **8.** $\sqrt{x^{2n}y^4}$

$$\sqrt{16 \times 25 \times 36} = 4 \times 5 \times 6 = 120$$

9. $\sqrt{25 \times 49 \times 121}$ **10.** $\sqrt{16 \times 25 \times 36 \times 144}$

11. $\sqrt[3]{27 \times 64 \times 125}$ **12.** $\sqrt[3]{8 \times 64 \times 216 \times 348}$

By the same principle, any root of a number may be found by resolving the number into its prime factors. Observe the following:

$$\sqrt{99225} = \sqrt{3^4 \cdot 5^2 \cdot 7^2} = 9 \cdot 5 \cdot 7 = 315$$

In like manner, solve:

13. $\sqrt{30625}$ **14.** $\sqrt{86436}$ **15.** $\sqrt[3]{21952}$ **16.** $\sqrt[3]{54872}$

Observe, also:

$$\sqrt[3]{45\cdot 60\cdot 80}=\sqrt[3]{(3^2\cdot 5)\cdot(2^2\cdot 3\cdot 5)\cdot(2^4\cdot 5)}$$
$$=\sqrt[3]{2^6\cdot 3^3\cdot 5^3}=60$$

Solve the following:

17. $\sqrt{14\times 21\times 42\times 63}$ **18.** $\sqrt{15a^5b^4\times 21b^2c^2\times 35a^3c^2}$

19. $\sqrt[3]{36\times 63\times 72\times 98}$ **20.** $\sqrt[3]{12a^7b^6\times 54b^3c^4\times 72a^5c^2}$

21. $\sqrt{(x^2+x-2)(x^2-x-6)(x^2-4x+3)}$

ROOT OF A FRACTION

312. From the law, $\left(\frac{a^m}{b^m}\right)^n=\frac{a^{mn}}{b^{mn}}$, we have—

$$\sqrt[n]{\frac{a^{mn}}{b^{mn}}}=\frac{a^m}{b^m}.$$

The **nth** *root of a fraction is the* **nth** *root of the numerator divided by the* **nth** *root of the denominator.*

Exercise 139

Give the following indicated roots:

1. $\sqrt{\frac{25a^4}{49b^6}}$ 2. $\sqrt[3]{\frac{1}{8}a^6}$ 3. $\sqrt[4]{\frac{16a^8b^{12}}{81x^4y^{16}}}$ 4. $\sqrt[3]{\frac{27a^{3n}b^6}{64x^{6n}y^9}}$

5. $\sqrt[3]{\frac{1}{27x^9}}$ 6. $\sqrt{\frac{1}{9}x^8}$ 7. $\sqrt[3]{\frac{216a^6b^9}{729x^3y^6}}$ 8. $\sqrt[4]{\frac{256a^4x^8}{625x^8y^4}}$

313. A root is called an **odd** root, if its index is an odd number; an **even** root, if its index is an even number.

NUMBER OF ROOTS

314. Since $8\times 8=64$, the square root of 64 is 8, and since $(-8)\times(-8)=64$, the square root of 64 is also -8.

To indicate that a root is positive or negative, the double sign, read *plus or minus*, is generally used:

$$\sqrt{a^4} = \pm a^2 \qquad \sqrt[4]{x^8} = \pm x^2 \qquad \sqrt{64} = \pm 8 \qquad \sqrt[4]{81} = \pm 3$$

321. Since odd powers have the same sign as the number involved, *odd roots have the same sign as the number. Thus,*

$$\sqrt[3]{8a^6} = 2a^2, \quad \sqrt[3]{-x^9} = -x^3, \quad \sqrt[5]{32a^5} = 2a, \quad \sqrt[5]{-243b^5} = -3b$$

322. The **principal** root of a number is the real root which has the same sign as the number itself.

The principal square root of 49 is 7, not -7. The principal cube root of 125 is 5; of -125 is -5.

TO FIND THE REAL ROOTS OF MONOMIALS

323. Rule.— *Find the required root of the coefficient, and divide the exponent of each letter by the index of the root.*

Give odd roots the sign of the number itself, and give even roots of positive numbers the double sign.

Exercise 140

Give the following roots:

1. $\sqrt[5]{32x^{10}}$	2. $\sqrt{\frac{4}{9}a^2b^4}$	3. $\sqrt{x^{2n}y^{4n}}$	4. $\sqrt[3]{-x^6y^3}$
5. $\sqrt[3]{27a^{12}}$	6. $\sqrt[3]{\frac{1}{8}x^6y^3}$	7. $\sqrt[n]{a^{3n}b^{3n}}$	8. $\sqrt[5]{-a^5b^5}$
9. $\sqrt[4]{81x^{12}}$	10. $\sqrt{\frac{9}{16}a^4c^8}$	11. $\sqrt[3]{x^{3n}y^{6n}}$	12. $\sqrt[3]{-x^6y^9}$

SQUARE ROOT OF A POLYNOMIAL

324. As we have learned, §§192 and 193, the square root of all trinomial squares, and the square root of some polynomial squares, may be determined by inspection.

We shall now show how to extract the square root of any polynomial square by the use of the following formula.

$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + 2\mathbf{ab} + \mathbf{b}^2$$

Since $(a+b)^2 = a^2 + 2ab + b^2$, the square root of the trinomial square is $a+b$. Comparing $a^2+2ab+b^2$ in this identity with its square root, we observe:

1. *The first term of the root is the square root of the first term of the arranged power.*

2. *If the square of the first term of the root is subtracted from the power, the remainder is* $2ab+b^2$.

The first term of the remainder is the product of twice the first term of the root and the second term. Therefore,

3. *The second term of the root is found by dividing the first term of the remainder by* $2a$.

$$2ab+b^2=(2a+b)b$$

4. *If we multiply the sum of* $2a$ *and* b *by* b *and subtract the result from* $2ab+b^2$, *the remainder is* 0.

The second member of this formula represents the square of any binomial; but since the terms of any polynomial may be grouped so as to form a binomial, $a^2+2ab+b^2$ may also represent the square of any polynomial.

If the root contains three terms, a^2 represents the square of a binomial, and $2ab$ represents twice the product of a *binomial* by a monomial; if the root contains four terms, a^2 represents the square of a *trinomial*, and $2ab$ represents twice the product of a *trinomial* by a monomial.

325. The following example illustrates the process of extracting the square root of a trinomial square.

$$\begin{array}{r|l} & 9a^6+12a^3x^2+4x^4 \;\big|\; \underline{3a^3+2x^2} \\ & 9a^6 \\ \hline 6a^3+2x^2 & \quad +12a^3x^2 \\ & \quad \underline{+12a^3x^2+4x^4} \end{array}$$

The first term of the root is $3a^3$, the square root of $9a^6$, which we place at the right of the trinomial square.

Subtracting the square of $3a^3$ from the trinomial, there remains a part that is represented in the formula by $2ab+b^2$.

Dividing the first term of the remainder by $6a^3$, we obtain the second term of the root, which is $2x^2$.

Multiplying $6a^3+2x^2$ $(=2a+b)$ by $2x^2$ $(=b)$, and subtracting the result from $12a^3x^2+4x^4$, there is no remainder.

From the trinomial we have subtracted the square of $3a^3$, twice the product of $3a^3$ and $2x^2$, the square of $2x^2$, and there is no remainder. $3a^3+2x^2$ is the square root of the trinomial.

In this work, the numbers represented by $2a$ and $2a+b$ are called respectively the *partial divisor* and the *complete divisor.*

Check: Calculate $(3a^3+2x^2)^2$ and compare the result with $9a^6+12a^3x^2+4x^4$.

326. We observe that in the extraction of the square root of a polynomial *subtraction* is an *essential process;* that is, the process consists in the subtraction from the polynomial of the parts of which the polynomial is composed. The first part subtracted is the square of the first term of the root, and the second part subtracted is a product, which the remainder is known to contain.

327. The same method applies to any polynomial whose root contains more than two terms.

If the root contains 3 terms, the subtraction of the square of the first term of the root, *which is a binomial,* is completed with the second subtraction. If the root contains 4 terms, the subtraction of the square of the first term of the root, *which is a trinomial,* is completed with the third subtraction; and so on.

The first partial divisor is twice a monomial; the second, twice a binomial; the third, twice a trinomial.

$$\begin{array}{lll}
 & 25a^4-40a^3x+46a^2x^2-24ax^3+9x^4 & \underline{\big|\,5a^2-4ax+3x^2} \\
 & \underline{25a^4} & \\
10a^2-4ax & \qquad\big|-40a^3x & \\
 & \qquad\underline{\big|-40a^3x+16a^2x^2} & \\
10a^2-8ax+3x^2 & \qquad\qquad\big|+30a^2x^2 & \\
 & \qquad\qquad\underline{\big|+30a^2x^2-24ax^3+9x^4} &
\end{array}$$

We find the first and second terms of the root as if we were getting the square root of a trinomial square.

Multiplying the first term of the root, $(5a^2-4ax)$, by 2, we have

9. $4x^8+40x^5-4x^6-16x^7+25x^4$

10. $64a^4-192a^3+64a^2+120a+25$

11. $25x^6+9x^2+1+10x^3-30x^4-6x$

12. $9a^4+6a^3b-47a^2b^2-16ab^3+64b^4$

13. $x^6-2x^5+5x^4-8x^3+8x^2-8x+4$

14. $264a+337a^2+81a^4+144+198a^3$

15. $16x^4+76x^2y^2+60xy^3+48x^3y+25y^4$

16. $36b^4+25a^4-30a^3b-36ab^3+69a^2y^2$

17. $9x^2-8x+16-10x^3-2x^5+x^6+3x^4$

18. $103a^2x^2+42a^3x+49a^4-48ax^3+64x^4$

19. $4a^2+25b^2+16c^2-20ab+16ac-40bc$

20. $4x^6+17x^4+10x^2-12x^5-4x-16x^3+1$

21. $25a^2-40ab+16b^2+70ax+49x^2-56bx$

22. $36c^4-60a^2c^2+25a^4-10a^2x^2+x^4+12c^2x^2$

23. $\frac{4a^4}{x^4}-\frac{4a^3}{x^3}-\frac{11a^2}{x^2}+\frac{6a}{x}+9$

24. $\frac{2a^4}{b^4}+\frac{a^2}{b^2}-\frac{2a^3}{b^3}+\frac{a^6}{b^6}-\frac{2a}{b}+1$

25. $x^4+\frac{2x^3}{a}+\frac{x^2}{a^2}+2ax+2+\frac{a^2}{x^2}$

26. $\frac{a^4}{b^4}-\frac{2a^2c}{b^2d}-\frac{3a^2}{2b^2}+\frac{c^2}{d^2}+\frac{3c}{2d}+\frac{9}{16}$

27. $\frac{a^4}{4}+a^2x^2-\frac{5a^2}{2}+x^4-5x^2+\frac{25}{4}$

SQUARE ROOT OF NUMBERS

329. The squares of the smallest and largest numbers of one, two, and three figures are as follows:

$1^2 = 1$	$10^2 = 1\ 00$	$100^2 = 1\ 00\ 00$
$9^2 = 81$	$99^2 = 98\ 01$	$999^2 = 99\ 80\ 01$

The number at the left of the sign in each identity is the square root of the number at the right.

It follows that if any square is separated into periods of two figures each, beginning at units, the number of figures in the root is the same as the number of periods.

When the number of figures in the square is odd, the left-hand period is incomplete, containing only one figure.

If a represents the tens and b the units in the square root of any square of three or four figures, $a+b$ represents the square root, and $a^2+2ab+b^2$ represents the square. Then the formula expresses this principle:

Any square of three or four figures is equal to the square of the tens of its square root, plus twice the product of the tens by the units, plus the square of the units.

For example,

$$57^2 = (50+7)^2 = 50^2 + 2(50\times 7) + 7^2 = 3249$$

$$\begin{array}{rr|l} & 54\ 76 & 70+4 \\ a^2 = & 49\ 00 & \\ \hline 2a = 140 & 5\ 76 & \\ 2a+b = 144 & 5\ 76 & \\ \hline \end{array}$$

Separating the number into periods of two figures each, we find that the root contains two figures, units and tens.

The square of the *number* of tens in the root is found wholly in 54. The largest square in 54 is 49, whose square root is 7. Hence, there are not more than 7 tens in the root.

Since there are 7 tens in the root, $a=70$, and $a^2=4900$. Subtracting a^2, which in this example is the square of 70, or 4900, from the number, we have a remainder of 576.

This remainder is the product of two factors, represented by $(2a+b)b$. The partial divisor, $2a$, is twice 70, or 140.

Dividing 576 by 140, the quotient is 4, which is probably the units' figure of the root. The complete divisor, $2a+b$, is 144.

Multiplying 144 by 4, and subtracting the product from 576, there is no remainder. Hence, $70+4$, or 74 is the root.

We may abbreviate and simplify the work somewhat by omitting the ciphers and condensing the other parts, as follows:

```
      54 76|74            22 09|47             96 04|98
      49                  16                   81
  144 | 5 76           87 | 6 09          188 | 15 04
      | 5 76              | 6 09              | 15 04
```

At first we write only 14, 8, and 18 of the partial divisors, and divide the remainder, *exclusive of the right-hand figure.*

If, on multiplying any complete divisor by the last figure of the root, the product is larger than the remainder, the last figure of the root is too large and must be diminished by 1.

After determining the units' figure of the root, we annex it to the partial divisor to form the complete divisor.

Exercise 142

Find the square root of the following:

1. 2304	2. 3481	3. 5184	4. 4761
5. 4624	6. 7396	7. 5776	8. 7569

330. The same method applies to any number whose root is expressed by more than two figures. It is only necessary to consider all the root already found as tens.

```
        57 15 36|756                 44 95 70 25|6705
        49                           36
   140 | 8 15                   127 | 8 95
       | 7 25                       | 8 89
  1506    | 90 36             13405    | 6 70 25
          | 90 36                      | 6 70 25
```

When the partial divisor is not contained in the dividend, exclusive of the right-hand figure, annex a cipher to the root and also to the divisor, and annex the next period to the dividend. In the second example above, 134 is not contained in 67.

333. Rule.— *Separate the decimal into periods of two figures each, beginning at tenths.*

The process is the same as with whole numbers.

From the right of the root point off as many decimal places as there are periods of decimal places.

Each period of a decimal must have two figures. If we wish the square root of a decimal to 2 places, we should have 4 decimal places in the number; if we wish to carry the work to 3 places, we should have 6 decimal places in the number; and so on. The number of decimal places may be increased by annexing ciphers.

Exercise 144

Find the approximate square root of the following:

1. 46.08	**2.** .4	**3.** .036	**4.** 5.826
5. 315.7	**6.** .8	**7.** .064	**8.** 95.25

TO FIND THE SQUARE ROOT OF A COMMON FRACTION

334. Rule.— *If both terms of a fraction are squares, find the square root of each term separately.*

If either term is not a square, reduce the fraction to a decimal, and find the square root of the decimal.

Exercise 145

Find the square roots of the following:

1. $4\frac{2}{5}$	**2.** $\frac{2}{3}$	**3.** $\frac{5}{6}$	**4.** $6\frac{2}{3}$
5. $7\frac{3}{4}$	**6.** $\frac{5}{7}$	**7.** $\frac{5}{8}$	**8.** $8\frac{5}{6}$
9. $6\frac{1}{2}$	**10.** $\frac{3}{4}$	**11.** $\frac{7}{8}$	**12.** $9\frac{3}{7}$

CHAPTER XXII

EXPONENTS. RADICALS

EXPONENTS

335. Fundamental Laws. Under certain restrictions the following laws have been established:

1. $\mathbf{a}^m \cdot \mathbf{a}^n = \mathbf{a}^{m+n}$ 2. $\mathbf{a}^m \div \mathbf{a}^n = \mathbf{a}^{m-n}$

3. $(\mathbf{a}^m)^n = \mathbf{a}^{mn}$ 4. $\sqrt[n]{\mathbf{a}^m} = \mathbf{a}^{\frac{m}{n}}$

5. $(\mathbf{ab})^n = \mathbf{a}^n\mathbf{b}^n$

The restrictions are that m and n are positive integers; in law 2 that m is greater than n, and in law 4 that m is exactly divisible by n.

336. But m and n might be 0, fractional, or negative numbers. The old definition that an exponent indicates how many times a number is taken as a factor can have no meaning for such exponents. We now extend the notion of exponent to give meanings to these new forms of exponent, but it is convenient to do this in such way that *the five laws above shall hold for the new forms* of exponent.

337. Definition of $\mathbf{a}^0$. In law 2, if m becomes equal to n, we have:

$$\frac{a^n}{a^n} = a^{n-n} = a^0. \text{ But, also } \frac{a^n}{a^n} = 1,$$

Therefore, $$a^0 = 1, \ (a = 0).$$

Any number (not itself 0) *with an exponent* 0 *equals* 1.

338. Definition of $\mathbf{a}^{\frac{m}{n}}$. In law 4, if m is not a multiple of n a fractional exponent arises. By the law of exponents for evolution we have:

$$a^{\frac{5}{2}} = \sqrt{a^5}, \quad a^{\frac{1}{2}} = \sqrt{a}, \quad a^{\frac{3}{4}} = \sqrt[4]{a^3}, \text{ and generally, } a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

A positive fractional exponent indicates a root of a power of the base. The denominator is the index of the root and the numerator is the exponent of the power.

339. Definition of a^{-n}. In law 2, if n is greater than m the quotient has a negative exponent.

Since law 1 is to hold for the new forms of exponent, we have: $$a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1$$

Therefore, $$a^{-n} \cdot a^n = 1$$

By the division axiom, § 15, $a^{-n} = \dfrac{1}{a^n}$

Any number with a negative exponent is equal to the reciprocal of the number with a numerically equal positive exponent.

Thus, $2a^{-1} = \dfrac{2}{a}$, $2^{-2}a^2 = \dfrac{a^2}{2_2}$, $a^3b^{-2}x^2y^{-3} = \dfrac{a^3x^2}{b^2y^3}$.

RADICALS

340. A **radical** is an indicated root of a number. **Roots** are indicated by the radical sign or by fractional exponents.

Thus,

$$\sqrt{a}+x,\ 5^{\frac{1}{3}},\ (a+b)^{\frac{1}{2}},\ \sqrt[3]{8a},\ a^{\frac{4}{5}},\ \text{and}\ \sqrt{x-4},$$

are all roots.

The **radicand** is the number whose indicated root is to be found. Thus the radicand of $\sqrt{15}$ is 15; of $\sqrt{9a}$ it is $9a$, and of $\sqrt{a-x}$, it is $a-x$.

In this chapter, except in § 375, it is to be understood that the sign $\sqrt{\ }$ means the ***positive*** *square root of the radicand.*

341. The **order, or degree,** of a radical is determined by the index of the root.

Thus, $\sqrt{29}$ is of the **second order, or second degree;** $\sqrt[3]{15}$ is a radical of the **third order** or **third degree.**

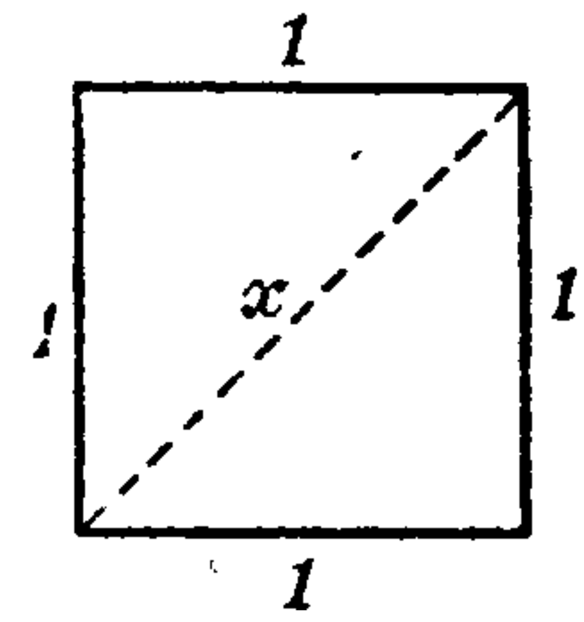
What is the degree of the root $a^{\frac{1}{4}}$? Of $x^{\frac{3}{5}}$?

342. A **rational number** is a positive or negative integer or a fraction whose terms are integers.

347. Surds arise in calculating, as the following 3 examples illustrate.

Exercise 146

1. Calculate the diagonal of a square whose sides are 1 unit long.

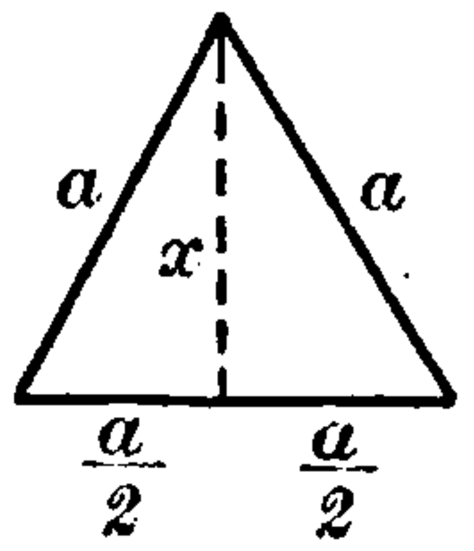


Letting x denote the length of the diagonal, we have

$$x^2=2, \text{ or } x=\sqrt{2}, \text{ which is a surd.}$$

2. Calculate the altitude of an equilateral triangle of side a.

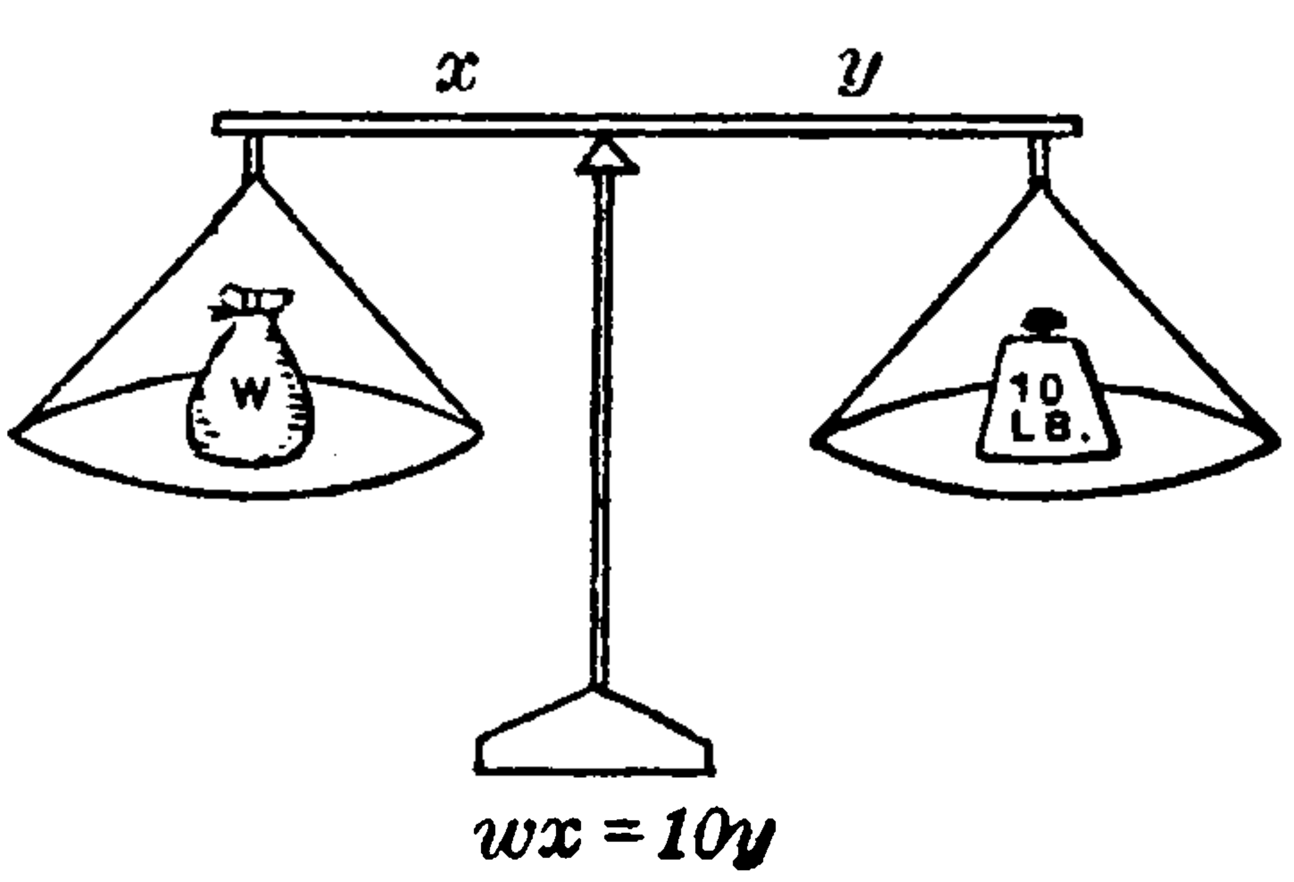
Letting x denote the length of the altitude, we have



$$x^2=a^2-\left(\frac{a}{2}\right)^2, \text{ or } x^2=\tfrac{3}{4}a^2,$$

which gives $x=\frac{a}{2}\sqrt{3}$, a mixed surd.

3. Determine the true weight of a body by means of a balance of unequal arms, x and y.



Let the true weight be denoted by w. When the body is placed in one pan suppose 10 lb. in the other pan just balance it. By the principle of the lever:

$$wx=10y \qquad (1)$$

When the body is placed in the other pan suppose 12 lb. just balance it. Then,

$$12x=wy. \qquad (2)$$

Dividing (1) by (2), we have $\frac{w}{12}=\frac{10}{w}$, and clearing,

$$w^2=120, \text{ or } w=\sqrt{120}.$$

We may also find the ratio of the unequal arms, by writing (2) thus,

$$wy=12x \qquad (3)$$

and dividing (1) by (3), obtaining

$$\frac{x}{y}=\frac{10y}{12x}$$

Multiplying through by $\frac{x}{y}$, $\qquad \frac{x^2}{y^2}=\frac{10}{12}.$

Extracting square roots, $\qquad \frac{x}{y}=\sqrt{\frac{10}{12}}$, which also is a surd.

SIMPLIFICATION OF RADICALS

348. The examples just given show the need for surds in calculating, that they arise just as other numbers arise in problem-solving, and that they are to be regarded as *numbers*.

349. Reduction of radicals is the process of changing their form without changing their value.

Radicals are simplified to get them into most convenient form for calculating.

A radical is not in its simplest form for calculating:

1. *If the radicand has a factor that is a power of the degree denoted by the index of the radical;*

2. *If the radicand is itself a power of the degree denoted by any factor of the index of the radical;*

3. *If there is a denominator under the radical sign, or a radical in any denominator.*

350. A radical may be simplified when the radicand has a factor whose indicated root can be found.

By the law of § 311, $\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$, also

$$\sqrt[3]{32a^4} = \sqrt[3]{8a^3 \cdot 4a} = \sqrt[3]{8a^3} \cdot \sqrt[3]{4a} = 2a\sqrt[3]{4a}$$

In all work in simplifying surds, only the *principal roots* are considered.

351. Rule.— *Take out of the radicand the largest factor whose indicated root can be found. Find this indicated root and write it as a coefficient of the other factor.*

Exercise 147

Simplify the following surds:

1. $\sqrt{12}$	2. $\sqrt{125}$	3. $\sqrt[3]{16}$	4. $\sqrt{16a^3b^3}$
5. $\sqrt{28}$	6. $\sqrt{288}$	7. $\sqrt[3]{56}$	8. $\sqrt{20x^2y^3}$
9. $\sqrt{32}$	10. $\sqrt{242}$	11. $\sqrt[3]{81}$	12. $\sqrt{12a^9b^2}$

The root of the rational factor, when found, is multiplied by the coefficient of the mixed surd.

Thus, $3\sqrt{108}=3\sqrt{36}\cdot\sqrt{3}=18\sqrt{3}$.

13. $2\sqrt{72}$ **14.** $\frac{1}{2}\sqrt{24}$ **15.** $a\sqrt{ab^3}$ **16.** $2a\sqrt[3]{3a^3}$

17. $3\sqrt{45}$ **18.** $\frac{1}{3}\sqrt{18}$ **19.** $x\sqrt[3]{x^3y}$ **20.** $3n\sqrt[3]{8n^4}$

21. $5\sqrt{48}$ **22.** $\frac{2}{3}\sqrt{63}$ **23.** $b\sqrt[3]{ab^4}$ **24.** $5a\sqrt[3]{5x^5}$

25. $(x+1)\sqrt{(x^2-1)(x-1)}$ **26.** $\sqrt[3]{(a+x)^2(a^2-x^2)}$

27. $(a+3)\sqrt{2a^2-12a+18}$ **28.** $\sqrt[3]{(y-x)^2(x^2-y^2)}$

352. From the principle of § 346 many roots can be calculated approximately from a few given values.

For example, given: $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt[3]{2}=1.260$, and $\sqrt[3]{3}=1.442$; to calculate other roots approximately,

$$\text{as } \sqrt{128}=8\sqrt{2}=8\cdot 1.414=11.312; \text{ and}$$
$$\sqrt[3]{250}=5\sqrt[3]{2}=5\cdot 1.732=8.66, \text{ etc.}$$

Exercise 148

From the given values of the square and cube roots of 2 and 3 calculate:

1. $\sqrt{8}$ **2.** $\sqrt{50}$ **3.** $\sqrt{98}$ **4.** $\sqrt{162}$

5. $\sqrt[3]{16}$ **6.** $\sqrt[3]{54}$ **7.** $\sqrt[3]{128}$ **8.** $\sqrt[3]{432}$

9. $\sqrt{12}$ **10.** $\sqrt{27}$ **11.** $\sqrt{75}$ **12.** $\sqrt{147}$

13. $\sqrt[3]{24}$ **14.** $\sqrt[3]{108}$ **15.** $\sqrt[3]{192}$ **16.** $\sqrt[3]{375}$

353. When the radicand is itself a power of the degree denoted by a *factor* of the index of the radical, proceed as shown in the following example:

$\sqrt[6]{25}=\sqrt[3]{\sqrt{25}}=\sqrt[3]{5}$, and $\sqrt[4]{49a^2b^4c^6}=\sqrt{\sqrt{7^2a^2b^4c^6}}$ $=\sqrt{7ab^2c^3}=bc\sqrt{7ac}$.

13. $\sqrt{\frac{5}{9}}$ 14. $\sqrt[3]{\frac{3}{4}}$ 15. $\sqrt[3]{\frac{8}{25}}$ 16. $\sqrt[4]{\frac{5}{27}}$

17. $\sqrt{\frac{3}{5}}$ 18. $\sqrt[4]{\frac{3}{8}}$ 19. $\sqrt[5]{\frac{2}{81}}$ 20. $\sqrt[5]{\frac{11}{16}}$

21. $\sqrt{\frac{3}{5}}$ 22. $\sqrt{\frac{8}{9}}$ 23. $\sqrt[6]{\frac{1}{32}}$ 24. $\sqrt[5]{\frac{10}{81}}$

25. $\sqrt{1-(\frac{1}{2})^2}$ 26. $\sqrt{2+(\frac{1}{3})^2}$ 27. $\sqrt[3]{1+(\frac{1}{2})^2}$

Exercise 151

Express with the radical sign and simplify:

1. $16^{\frac{2}{3}}$ 2. $(4a)^{\frac{5}{3}}$ 3. $x^{\frac{3}{2}}y^{\frac{5}{2}}$ 4. $24^{\frac{1}{3}}n^{\frac{4}{3}}$

5. $27^{\frac{2}{5}}$ 6. $x^{\frac{5}{3}}y^{\frac{7}{3}}$ 7. $a^{\frac{4}{3}}b^{\frac{7}{3}}$ 8. $75^{\frac{1}{2}}x^{\frac{5}{2}}$

TO REDUCE A MIXED NUMBER TO AN ENTIRE SURD

357. A **mixed surd** may be reduced to an **entire surd** by reversing the process of simplifying surds. Thus,

$$5\sqrt{8}=\sqrt{25\cdot 8}=\sqrt{200}, \text{ and } 3\sqrt[3]{5}=\sqrt[3]{27\cdot 5}=\sqrt[3]{135}$$

Exercise 152

Express the following as entire surds:

1. $2\sqrt[3]{7}$ 2. $\frac{2}{3}\sqrt{27}$ 3. $2a\sqrt{5a}$ 4. $1\frac{1}{2}a\sqrt{8a^2x}$

5. $5\sqrt{3}$ 6. $\frac{1}{2}\sqrt[4]{48}$ 7. $2x\sqrt[3]{3x}$ 8. $1\frac{1}{3}x\sqrt{9x^3y}$

9. $2\sqrt[4]{5}$ 10. $\frac{3}{2}\sqrt[3]{32}$ 11. $5a\sqrt[n]{5a}$ 12. $2\frac{1}{2}a\sqrt{8ax^2}$

13. $\frac{3}{2}\sqrt[3]{\frac{4}{9}}$ 14. $\frac{2}{5}\sqrt{\frac{25}{27}}$ 15. $\frac{3b}{2a}\sqrt[3]{\frac{8a}{9b}}$ 16. $\frac{1\frac{1}{4}b}{1\frac{1}{8}a}\sqrt[3]{\frac{5ax^2}{8bc^3}}$

17. $\frac{a-2}{a+2}\sqrt{\frac{a+2}{a+2}}$ 18. $(a+x)\sqrt{\frac{a-x}{a+x}}$ 19. $\frac{x+4}{x-4}\sqrt{\frac{x+4}{x-4}}$

ADDITION AND SUBTRACTION OF SURDS

358. Surds are added or subtracted by adding or subtracting their **coefficients**.

359. Similar surds are surds which in their simplest form are of the same degree and have the same radicand, such as:

$$2\sqrt{5},\ 4\sqrt{5},\ a\sqrt{5};\ a\sqrt{x},\ b\sqrt{x};\text{ and}$$
$$3\sqrt[3]{7},\ 5\sqrt[3]{7},\ 9\sqrt[3]{7},\text{ etc.}$$

Two or more surds can be united into one by addition or subtraction only when they are similar, as is shown here:

$$2\sqrt{45}+4\sqrt{20}+5\sqrt{80}-3\sqrt{125}=$$
$$6\sqrt{5}+8\sqrt{5}+20\sqrt{5}-15\sqrt{5}=19\sqrt{5},$$

and

$$5\sqrt[3]{16x^2}-\sqrt[3]{54x^2}-\sqrt[3]{2x^2}=$$
$$10\sqrt[3]{2x^2}-3\sqrt[3]{2x^2}-\sqrt[3]{2x^2}=6\sqrt[3]{2x^2}$$

Exercise 153

Simplify the following:

1. $3\sqrt{300}+2\sqrt{243}$
2. $2\sqrt[3]{16x^2}-\sqrt[3]{54x^3}+\sqrt[3]{250x^3}$
3. $4\sqrt{45}+2\sqrt{48}-4\sqrt{27}-3\sqrt{20}+2\sqrt{12}$
4. $7\sqrt{175}-5\sqrt{112}$
5. $4\sqrt{16a^3}-2\sqrt{25a^3}+2\sqrt{36a^3}$
6. $3\sqrt{112}+6\sqrt{45}-3\sqrt{28}-\frac{3}{4}\sqrt{80}+3\sqrt{63}$
7. $3\sqrt[3]{375}-2\sqrt[3]{192}$
8. $3\sqrt{24a^5}-\sqrt{96a^5}+2\sqrt{54a^5}$
9. $2\sqrt{360}-4\sqrt{10}-\frac{2}{3}\sqrt{90}+3\sqrt{40}-\frac{1}{5}\sqrt{250}$
10. $4\sqrt{128}-3\sqrt{162}$
11. $2\sqrt[4]{81x^5}-3\sqrt[4]{16x^5}+\sqrt[4]{80x^5}$
12. $5\sqrt[3]{16}+\frac{1}{2}\sqrt[3]{128}-5\sqrt[3]{54}+4\sqrt[3]{250}-2\sqrt[3]{486}$

TO REDUCE SURDS TO THE SAME ORDER

360. Surds of different orders are changed to the same order *by expressing the radicals as fractional exponents, and reducing the fractional exponents to equivalent fractions having a common denominator, and then expressing the surds with radical signs.* Thus,

$$\sqrt{2}=2^{\frac{1}{2}},\text{ and }\sqrt[3]{3}=3^{\frac{1}{3}}.$$

The lowest common denominator of the exponents is 6.

Then, $2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}$, and

$3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}$.

Since $\sqrt{2} = \sqrt[6]{8}$ and $\sqrt[3]{3} = \sqrt[6]{9}$, we can at once say that

$$\sqrt[3]{3} > \sqrt{2}.$$

This principle enables us to compare radicals of different orders as to *relative magnitude.*

The **signs of inequality** are $>$ and $<$. The sign, $>$, means *greater than*; $<$ means *less than.*

Exercise 154

Compare the following pairs of radicals:

1. $\sqrt{5}$ and $\sqrt[3]{7}$
2. $\sqrt[5]{5}$ and $\sqrt{2}$
3. $2\sqrt{3}$ and $3\sqrt[3]{2}$
4. $\sqrt[3]{4}$ and $\sqrt[4]{6}$
5. $\sqrt[3]{3}$ and $\sqrt[5]{6}$
6. $2\sqrt{5}$ and $3\sqrt[3]{3}$
7. Arrange in order of value, $\sqrt[4]{7}$, $\sqrt[6]{6}$, and $\sqrt{2}$.

MULTIPLICATION OF SURDS

361. The product of two or more surds of the same order is found by law 5, § 335.

For *fractional exponents* this law takes the form:

$$\sqrt[n]{\mathbf{a}} \cdot \sqrt[n]{\mathbf{b}} = \sqrt[n]{\mathbf{ab}}$$

Notice that this applies only when the surds are of the same order. Thus,

$$\sqrt{5} \cdot \sqrt{35} = \sqrt{175} = 5\sqrt{7}, \text{and}$$
$$2\sqrt[3]{6} \cdot 3\sqrt[3]{18} = 6\sqrt[3]{108} = 18\sqrt[3]{4}$$

Exercise 155

Multiply as indicated:

1. $4\sqrt{3} \cdot 3\sqrt{5}$
2. $2\sqrt{7} \cdot 3\sqrt{7}$
3. $2\sqrt{5} \cdot 3\sqrt{15}$
4. $5\sqrt[3]{2} \cdot 2\sqrt[3]{4}$
5. $4\sqrt{5} \cdot 5\sqrt{5}$
6. $5\sqrt[3]{2} \cdot 3\sqrt[3]{32}$
7. $3\sqrt[4]{4} \cdot 4\sqrt[4]{8}$
8. $3\sqrt{6} \cdot 2\sqrt{8}$
9. $4\sqrt[3]{6} \cdot 2\sqrt[3]{12}$

Exercise 157

Multiply the following:

1. $3-\sqrt{a}$ by $4+2\sqrt{a}$
2. $5+4\sqrt{2}$ by $3-3\sqrt{2}$
3. $12+3\sqrt{5}$ by $4-2\sqrt{5}$
4. $3a-3\sqrt{a}$ by $2a+2\sqrt{a}$
5. $\sqrt{3}-2\sqrt{6}$ by $\sqrt{3}+5\sqrt{6}$
6. $2\sqrt{a}+3\sqrt{c}$ by $\sqrt{a}+6\sqrt{c}$
7. $\sqrt{2}+\sqrt{3}-\sqrt{5}$ by $\sqrt{2}-\sqrt{3}+2\sqrt{5}$
8. $\sqrt{3}-\sqrt{5}+2\sqrt{7}$ by $2\sqrt{3}+2\sqrt{5}-\sqrt{7}$
9. $3\sqrt{7}-2\sqrt{3}+4\sqrt{5}$ by $4\sqrt{7}-3\sqrt{3}-\sqrt{5}$

Multiply by inspection:

10. $(\sqrt{7}+2)(\sqrt{7}-2)$
11. $(\sqrt{a}+\sqrt{x})(\sqrt{a}+\sqrt{x})$
12. $(\sqrt{3}+5)(\sqrt{3}+5)$
13. $(\sqrt{x}-\sqrt{y})(\sqrt{x}-\sqrt{y})$
14. $(x-\sqrt{8})(x-\sqrt{8})$
15. $(\sqrt{8}-\sqrt{3})(\sqrt{8}+\sqrt{3})$
16. $(\sqrt{a}+x)(\sqrt{a}-x)$
17. $(\sqrt{x}+\sqrt{5})(\sqrt{x}+\sqrt{5})$
18. $(\sqrt{10}-5)(\sqrt{10}-5)$
19. $(\sqrt{8}+\sqrt{3})(\sqrt{8}+\sqrt{3})$

DIVISION OF SURDS

364. The quotient of two surds of the same order is found by the **principle of evolution** as stated in the formula:

$$\frac{\sqrt[n]{\mathbf{x}}}{\sqrt[n]{\mathbf{y}}}=\sqrt[n]{\frac{\mathbf{x}}{\mathbf{y}}}$$

As in multiplication, so here, this principle applies only to surds of the *same order*. Thus,

$$2\sqrt{60} \div \sqrt{5} = 2\sqrt{12} = 4\sqrt{3},$$

$$\sqrt{54} \div 3\sqrt{3} = \tfrac{1}{3}\sqrt{18} = \sqrt{2},$$

also

$$6\sqrt{15} \div 2\sqrt{18} = 3\sqrt{\tfrac{5}{6}} = \tfrac{1}{2}\sqrt{30}.$$

Exercise 158

Give these quotients by inspection:

1. $4\sqrt{24} \div \sqrt{3}$
2. $\sqrt{32} \div 2\sqrt{2}$
3. $6\sqrt{45} \div 2\sqrt{3}$
4. $5\sqrt{81} \div \sqrt{7}$
5. $\sqrt{54} \div 3\sqrt{2}$
6. $2\sqrt{90} \div 9\sqrt{5}$
7. $3\sqrt{40} \div \sqrt{2}$
8. $\sqrt{56} \div 2\sqrt{7}$
9. $3\sqrt{75} \div 5\sqrt{3}$

RATIONALIZING SURDS

365. Rationalizing is the process of multiplying a surd by a number that gives a rational product. Observe the following:

$$\sqrt{5} \cdot \sqrt{5} = 5, \qquad \sqrt[3]{3} \cdot \sqrt[3]{9} = 3,$$

$$\sqrt{8} \cdot \sqrt{2} = 4, \quad \text{and} \quad \sqrt[3]{25} \cdot \sqrt[3]{5} = 5.$$

The rationalizing factor is the factor by which a surd is multiplied to give a rational product.

When the product of two surds is rational, either surd is the rationalizing factor of the other.

Name a rationalizing factor of each of the following surds and give the products:

1. $\sqrt{6}$
2. $2\sqrt{12}$
3. $5\sqrt[3]{12}$
4. $2\sqrt{27ab}$
5. $\sqrt{8}$
6. $5\sqrt[3]{32}$
7. $3\sqrt[3]{18}$
8. $4\sqrt[3]{16xy}$

366. A **binomial surd** is a binomial *one* or *both* of whose terms are surds. Thus, $4+\sqrt{5}$, $\sqrt{3}-2$, and $\sqrt{6}+\sqrt{7}$.

367. **A binomial quadratic surd** is a binomial surd whose surd, or surds, are of the *second order*.

368. **Conjugate surds** are two binomial quadratic surds that differ only in the sign of one of the terms.

For example $a+\sqrt{b}$ and $a-\sqrt{b}$, as also

$$\sqrt{7}-\sqrt{5} \text{ and } \sqrt{7}+\sqrt{5}$$

are conjugate surds.

Since conjugate surds are of forms $a+b$ and $a-b$, *the product of any two conjugate surds is rational.*

Hence it follows that *any binomial quadratic surd may be rationalized by multiplying it by its conjugate.*

Thus, $(4+\sqrt{7})(4-\sqrt{7})=9$,

and $(\sqrt{10}-\sqrt{2})(\sqrt{10}+\sqrt{2})=8$

Exercise 159

Name a rationalizing factor of each of the following surds and give the products:

1. $8-\sqrt{14}$
2. $2\sqrt{5}-3\sqrt{2}$
3. $\sqrt{a+x}+\sqrt{a}$
4. $a+2\sqrt{b}$
5. $2\sqrt{15}+\sqrt{7}$
6. $\sqrt{a}-\sqrt{a-b}$
7. $\sqrt{75}-5$
8. $\sqrt{70}-3\sqrt{6}$
9. $\sqrt{a-x}+\sqrt{a}$

Exercise 160

Rationalize the denominators of the following:

1. $\dfrac{4}{3-\sqrt{5}}$
2. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$
3. $\dfrac{\sqrt{a+1}+2}{\sqrt{a+1}-2}$
4. $\dfrac{3-\sqrt{2}}{3+\sqrt{2}}$
5. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$
6. $\dfrac{a-\sqrt{x+1}}{a+\sqrt{x+1}}$
7. $\dfrac{a}{a-\sqrt{b}}$
8. $\dfrac{\sqrt{a}-\sqrt{x}}{\sqrt{x}-\sqrt{a}}$
9. $\dfrac{\sqrt{a+b}+c}{\sqrt{a+b}-c}$
10. $\dfrac{a-\sqrt{x}}{a+\sqrt{x}}$
11. $\dfrac{\sqrt{a+b}+c}{\sqrt{a+b}-c}$
12. $\dfrac{\sqrt{x^2-9}+3}{\sqrt{x^2-9}-3}$

7. $12+4\sqrt{5}$ 8. $15+3\sqrt{6}$ 9. $30-6\sqrt{20}$

10. $2x+3y-2\sqrt{6xy}$ 11. $2x+2\sqrt{x^2-y^2}$

12. $2a+b-2\sqrt{a^2+ab}$ 13. $a^2+b+2a\sqrt{b}$

APPROXIMATE VALUES OF SURDS

372. The **approximate value of a surd** is found by extracting the indicated root to the required degree of accuracy. It is frequently necessary to find the value of a fraction with a radical denominator.

In such work much labor is saved by first rationalizing the *divisor*, or *denominator*. Thus,

$$\frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}=\frac{3\cdot 2.23607}{5}$$

Simplify each of the following divisions, finding the numerical value correct to 5 decimal places, having given that,

$$\sqrt{2}=1.41421,\quad \sqrt{3}=1.73205,\ \text{and}\ \sqrt{5}=2.23607.$$

Exercise 162

1. $10\div\sqrt{18}$ 2. $13\div\sqrt{27}$ 3. $25\div\sqrt{20}$

4. $3\sqrt{15}\div 2\sqrt{3}$ 5. $18-3\sqrt{8}$ 6. $7\div 2\sqrt{75}$

IRRATIONAL EQUATIONS IN ONE UNKNOWN

373. An **irrational**, or **radical equation** is an equation containing an **irrational** root of the **unknown number**. Thus,

$$\sqrt{x}=3,\quad \sqrt{x-4}=5,\quad \sqrt{3x-5}=\sqrt{x+35}.$$

To solve an irrational equation the first step is to free the equation of radicals. This is done by raising both members of the equation to the same power.

Power Axiom.— *The same powers of equal numbers are equal.*

To solve, $\sqrt{2x-5}=3$, or $\sqrt{5x-7}=\sqrt{2x+17}$.

Squaring, $2x-5=9$, $5x-7=2x+17$.

The results of squaring in these two examples are simple equations, and are solved as such.

Radical equations containing more than one radical may have to be squared more than once.

Thus, to solve:

$$\sqrt{x-5}+\sqrt{x}=5$$

Subtracting $\sqrt{x}$, $\sqrt{x-5}=5-\sqrt{x}$

Squaring, $x-5=25-10\sqrt{x}+x$

Uniting terms, $10\sqrt{x}=30$

Dividing by 10, $\sqrt{x}=3$

Squaring, $x=9$.

374. With radical equations it is agreed that the radical sign shall denote only *principal roots.*

Verifying, $\sqrt{9-5}+\sqrt{9}=5$

$$2+3=5$$

$$5=5$$

Since the substitution of 9 for x in the original equation gives an identity, 9 satisfies the equation.

Exercise 163

Solve and verify the following:

1. $\sqrt{x+4}=9$

2. $\sqrt{x-a}=a$

3. $\sqrt{x+5}=4$

4. $\sqrt[3]{x+6}=2$

5. $\sqrt{x+b}=a$

6. $\sqrt[3]{x-3}=3$

7. $\sqrt{x}-\sqrt{x-5}=\sqrt{5}$

8. $\sqrt{x+2}+\sqrt{x}=2$

9. $\sqrt{x-7}+\sqrt{7}=\sqrt{x}$

10. $\sqrt{x+5}-\sqrt{x}=1$

11. $\sqrt{x+9}-\sqrt{x-7}=2$

12. $\sqrt{x-4}+\sqrt{x}=2$

13. $3\sqrt{x}-\sqrt{4x-9}=\sqrt{3}$

14. $\sqrt{x}+\sqrt{x+8}=4$

15. $\dfrac{\sqrt{3x+9}+5\sqrt{x-5}}{\sqrt{3x+9}-5\sqrt{x-5}}=8$

16. $\dfrac{\sqrt{x+9}}{\sqrt{x+3}}=\dfrac{\sqrt{x+2}}{\sqrt{x-2}}$

17. $\dfrac{10}{\sqrt{x-5}}=\sqrt{x}+\sqrt{x-5}$ **18.** $\dfrac{\sqrt{x+6}}{\sqrt{x-3}}=\dfrac{\sqrt{x+2}}{\sqrt{x-4}}$

19. $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{x+a}}$ **20.** $\sqrt{a-\sqrt{x}}+\sqrt{x+a}=\sqrt{x}$

21. $\sqrt{x+5}+\sqrt{x-2}=\sqrt{5+4x}$

375. A statement may be in the *form* of an irrational equation which, under the assumption that $\sqrt{\ }$ shall mean the *positive* square root, cannot be satisfied.

Thus, solving by the usual method,

$$\sqrt{x-5}=\sqrt{x}+5$$

we obtain $x=9$. Attempting to verify we have

$$2=3+5$$

which is *not an identity.*

Setting aside the assumption and recalling that $\sqrt{x}$ may be either the *positive* or *negative* root, *as the conditions of the problem require*; and retaining both signs in verifying, we have,

$$\pm 2=\pm 3+5.$$

Of these possibilities as to sign, we can get an identity by using $+2$ for $\sqrt{x-5}$ and -3 for $\sqrt{x}$. It is worth noting that this state of things would not have been found if verifying had been omitted.

In squaring a radical equation, a root is sometimes *introduced* which the given equation did not contain. Thus,

$$\sqrt{4x+1}=3-\sqrt{x-2}$$

freed of radicals and solved by the usual process, leads to

$$x=2 \text{ and } x=6$$

Verifying for 2, $3=3-0$ This checks.

Verifying for 6, $5=3-2$ This does not check.

Hence, 2 satisfies the equation under the assumption that $\sqrt{\ }$ indicates only the *positive square root,* while 6 does not

CHAPTER XXIII

QUADRATIC EQUATIONS

377. A **quadratic equation** is an equation of the second degree in the unknown number. For example,

$$x^2+5x=20,\ 4x^2=36, \text{ and } 2x^2-4x=5a,$$

are all quadratic equations.

In determining the degree of an equation it is assumed that the equation is first reduced to its simplest form.

378. The **constant term** in a quadratic equation is the term that does not contain the unknown number.

Some quadratic equations contain only the square of the unknown number; others contain both the square and the first power of it. Hence there are two kinds of quadratic equations.

379. A **pure quadratic equation** is an equation that does not contain the first power of the unknown number. Thus,

$$3x^2=108,\ x^2+2x=2x^2+2x-16,\ 4x^2=36a.$$

380. An **affected quadratic equation** is an equation that contains both the first and second powers of the unknown number. Thus,

$$3x^2+5x=15,\ x^2-4x=8,\ x^2-ax=b.$$

Pure quadratics are also called **incomplete quadratics,** and affected quadratics are called **complete quadratics.**

THE GRAPHICAL METHOD OF SOLUTION

381. The Graphical Solution. The **normal form** of the *pure,* or *incomplete quadratic,* is $x^2-a=0$.

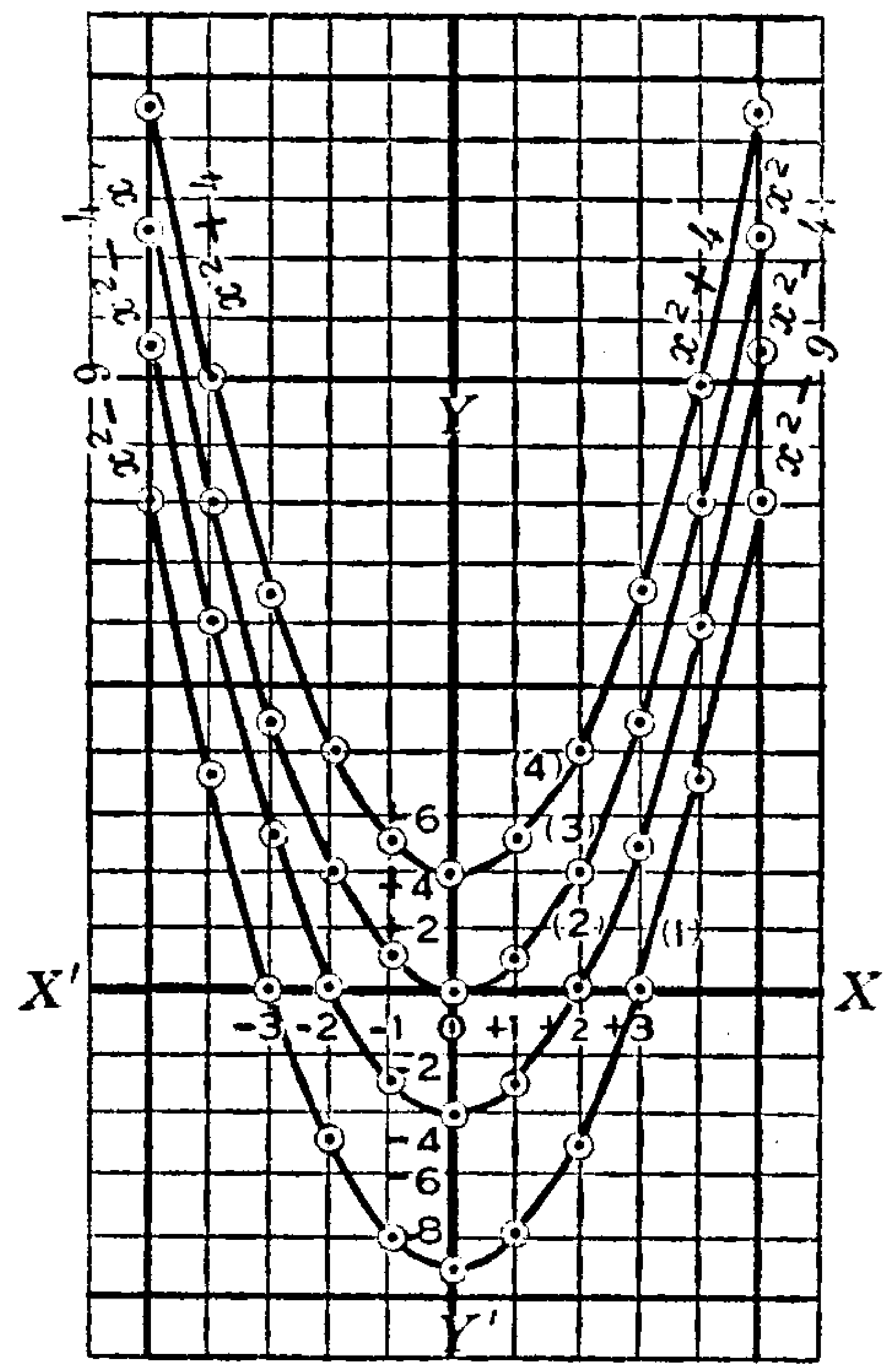
Exercise 164 — Graphing

We shall now graph x^2-a, for $a=9$, $a=4$, $a=0$, and $a=-4$.

1. Graphing x^2-a for $a=9$, or graphing x^2-9, we first calculate and locate the points:

$x=$	0,	1,	2,	3,	4,	5,	−1,	−2,	−3,	−4,	−5
$x^2-9=$	−9,	−8,	−5,	0,	7,	16,	−8,	−5,	0,	+7,	+16

Draw a smooth curve (1) through these points. Recall that $x^2-9=0$ asks: "What is x where x^2-9 is 0?" or "What is x where the curve *crosses the horizontal*?" The answer is readily seen from the figure to be +3 and −3.



Scale
1 = 1 horizontal space
2 = 1 vertical space

Hence, the **roots of $x^2-9=0$** are +3 and −3. These substituted in $x^2-9=0$ are seen to satisfy it.

2. Graphing x^2-a for $a=4$, or graphing x^2-4, we calculate and plot the points:

$x=$	0,	1,	2,	3,	4,	5,	−1,	−2,	−3,	−4,	−5
$x^2-4=$	−4,	−3,	0,	+5,	+12,	+21,	−3,	0,	+5,	+12,	+21

and draw a smooth curve, like curve (2), through the points.

This curve is of the same form as curve (1), but is simply raised upward 5 units. The x-values of the crossing points are here +2 and −2, which are the **roots of $x^2-4=0$.**

3. Similarly, graphing **curve (3) for x^2-a**, for $a=0$, or graphing the curve for x^2, the required curve is drawn through the following calculated and plotted points:

$x=$	0,	1,	2,	3,	4,	5,	−1,	−2,	−3,	−4,	−5
$x^2=$	0,	1,	4,	9,	16,	25,	1,	4,	9,	16,	25

Here there is but *one* x-value of the crossing-, or rather touching-point with the horizontal, viz.: 0.

Because there were two crossing-points as the curve moved upward *so long as it crossed the horizontal,* we say there are *two equal* 0's here. In reality there is only the root 0, because $+0$ and -0 are the same point.

4. Graphing x^2-a for $a=-4$, or graphing x^2+4, we calculate and plot the points:

$$\begin{array}{rcccccccccccc} x= & 0, & 1, & 2, & 3, & 4, & 5, & -1, & -2, & -3, & -4, & -5 \\ x^2+4= & +4, & +5, & +8, & +13, & +20, & +29, & +5, & +8, & +13, & +20, & +29, \end{array}$$

and draw the smooth curve (4) through them. The curve being 4 *units higher than curve* (3) does not touch the horizontal at all. There are no crossing-points and the algebraic way of saying this is to say the roots are *imaginary.* We shall see later that the roots are $+2\sqrt{-1}$ and $-2\sqrt{-1}$.

382. We see then that a pure quadratic in general has *two roots* that are numerically *equal* but of *opposite* signs, but that if the graph of the first member just touches the horizontal there is but *one* root, viz., 0. If the graph does not cut the horizontal, there are *no real roots.*

But since two results are found by solving

$$x^2=-a$$

i.e., $$x=\sqrt{-a}, \text{ and } x=-\sqrt{-a},$$

we say that if the graph lies *entirely above the horizontal,* there are *two* roots, *one positive* and the *other negative,* and both *imaginary.*

SOLVING QUADRATICS BY FACTORING

383. The solution of quadratic equations *by factoring,* given in § 215 and on page 164, should be reviewed here.

This is not a general method, for it is limited to those equations the first members of which are readily factored.

A pure quadratic equation which is reducible to the form $x^2-a=0$ is readily solved by factoring.

When reduced to this form it is evident that the first

Exercise 166

Solve the following by factoring and verify:

1. $x^2+11x-26=0$
2. $4x^2-12x=-9$
3. $2x^2-5x-12=0$
4. $6x^2+11x=-4$
5. $3x^2-7x-20=0$
6. $x^2-20x=-51$

385. Some equations of a *higher degree than the second* may be solved by factoring. Observe the following:

$$x^3-x^2=12x$$
$$x(x-4)(x+3)=0$$
$$x=0,\ 4,\ \text{and}\ -3$$

$$x^3-x^2-4x+4=0$$
$$(x-2)(x+2)(x-1)=0$$
$$x=2,\ -2,\ \text{and}\ 1.$$

Substitute these values of the unknown in the given equations from which they were found, and verify that they are the correct roots.

Exercise 167

Solve the following by factoring and verify:

1. $x^3+8x^2-9x=0$
2. $x^3+5x^2-x=5$
3. $x^2+ab-ax-bx=0$
4. $x^4-5x^2+4=0$
5. $x^3+x^2-42x=0$
6. $x^2+ax+bx+ab=0$
7. $x^3+5x-6x^2=0$
8. $6x^2-49x=-8$
9. $x(x^2-1)-2(x+1)=0$
10. $x^3+x^2-30x=0$
11. $x^3+7x^2-7=x$
12. $x(x^2-4)-3(x-2)=0$
13. $6x^2+3x-18=0$
14. $x^3-x^2+9=9x$
15. $(x-2)^2-4(x-2)+3=0$
16. $x^3+5x^2-6x=0$
17. $6x^2+17x=-5$
18. $(x^2-x-2)(3x^2-x-2)=0$
19. $6x^2-5x-21=0$
20. $x^4-17x^2+16=0$

SQUARE ROOT METHOD OF SOLUTION

386. A pure quadratic is solved by reducing it to the **normal form,** $x^2=a$, and taking the square root of both members.

Root Axiom. *Equal principal roots of equal numbers are equal.*

Extracting the square root of both members, we have:

$$x=\pm\sqrt{a}$$

The double sign belongs to the unknown number as well as to the second member, but $x=\pm\sqrt{a}$ is the same as $-x=\pm\sqrt{a}$. For this reason the double sign is used before the *second member* only.

A pure quadratic equation has two roots numerically equal, one positive and the other negative.

For example,

$x^2=25$,	$x^2=8$,	$x^2=-5$, have the roots:
$x=\pm 5$,	$x=\pm 2\sqrt{2}$,	$x=\pm\sqrt{-5}$.

Since the square root of a negative number is imaginary, we observe that when a is negative, both roots are imaginary.

All this was shown more clearly in § 381 by the aid of the graphs.

Exercise 168

Solve by the square root method:

1. $\frac{1}{2-x}+\frac{1}{2+x}=5$

2. $\frac{3}{x-4}-\frac{3}{x+4}=\frac{1}{2}$

3. $\sqrt{x+5}=\frac{5}{\sqrt{x-5}}$

4. $\frac{a}{\sqrt{x+a}}=\sqrt{x-a}$

387. Any **complete quadratic equation** may be reduced to the *normal form,*

$$\mathbf{ax^2+bx+c=0,}$$

a, b, and c denoting any real numbers, positive or negative, integral or fractional, though a may not be 0.

Since any complete quadratic may be reduced to this form, it is called the **general quadratic.**

To apply the *square root method* of solution, the first member must be made a square. For this purpose the form of the equation is changed to:

$$ax^2+bx=-c$$

388. The process of making the first member of a quadratic equation a square is called **completing the square.**

The value of a in the general quadratic, ax^2+bx+c, may be 1, or it may be any number greater than 1.

TO COMPLETE THE SQUARE WHEN a IS 1

389. Consider the arranged **trinomial square,**

$$x^2+2cx+c^2.$$

Two of the terms are squares and the other term is the product of three factors, viz.: The factor 2, the square root of the *first term*, and the square root of the *last term.*

The binomial x^2+2cx represents the sum of the first and second terms of *any arranged trinomial square.* Dividing the second term, $2cx$, by twice the square root of the first term, *i.e.*, by $2x$, the quotient is c, which is the square root of the missing term. Adding c^2 to x^2+2cx will therefore *complete the square.*

390. Rule.— *Reduce the equation to the general form and add to both members the square of half the coefficient of x.*

To make the first member of $x^2-6x=7$ a square, we must add 9 to both members, thus obtaining:

$$x^2-6x+9=16$$

By the root axiom, $x-3=\pm 4$

Whence, $x=7$, and -1

Substitute these in the given equation and verify.

Carefully observe the following important truth:

The *sum* of the roots must be the negative *coefficient of* x in the equation in which the coefficient of x^2 is 1 [*i.e.*, in (1)], and the *product* of the roots must be the *constant* term in the same equation.

The sum of the roots is 3, the coefficient of x with reversed sign; the product is $-\frac{5}{2}$, which is the constant term.

Exercise 170

Solve the following and verify:

1. $x^2-168=-2x$	**2.** $2x^2+3x-14=0$
3. $3x^2-10x=-3$	**4.** $3x^2+4x-39=0$
5. $y^2-120=-2y$	**6.** $2x^2+7x-39=0$
7. $8x-x^2=-180$	**8.** $n^2-11n-60=0$
9. $x^2-16x=-60$	**10.** $y^2+15y-54=0$
11. $3x^2-33=-2x$	**12.** $x^2-13x-30=0$
13. $n^2-11n=-30$	**14.** $3x^2+x-200=0$
15. $3x^2-95=-4x$	**16.** $y^2-11y+28=0$

393. To avoid fractions, first multiply both members of the equation by *four times the coefficient of* x^2.

For example, to solve: $2x^2-7x-15=0$

Multiply by 8, $16x^2-56x=120$,

Dividing $56x$ by twice the square root of $16x^2$, the quotient is 7. Squaring 7 and adding,

$$16x^2-56x+49=169$$

By the root axiom, $4x-7=\pm 13$,

Whence, $x=5$ and $-\frac{3}{2}$.

If the coefficient of x^2 in the given equation is made 1, the coefficient of x is $-\frac{7}{2}$ and the constant term is $-\frac{15}{2}$.

The sum of the roots is $\frac{7}{2}$, the coefficient of x with reversed sign; the product is $-\frac{15}{2}$, which is the constant term. This checks the work.

Observe that the number added to complete the square is the *square of the coefficient of x in the given equation.*

Exercise 171

Complete the square, solve and verify:

1. $3x^2-7x=-2$
2. $2x^2-5x-42=0$
3. $x^2-12=-4x$
4. $3x^2-2x-40=0$
5. $4x^2-7x=-3$
6. $5r^2-14r+8=0$
7. $m^2+6m=-11$
8. $3u^2+9u-30=0$
9. $2n^2-5=-3n$
10. $3y^2-10y+3=0$
11. $x^2+6x=-25$
12. $y^2-10y+21=0$
13. $3t^2-2=-5t$
14. $2s^2+7s-22=0$
15. $a^2+8a=-21$
16. $B^2-12B-45=0$

SOLUTION BY FORMULA

394. The equation $\mathbf{ax^2+bx+c=0}$ may be taken to represent, or typify, *any quadratic equation*, in which all terms have been transposed to the first member, the x^2-terms being combined into a single term, as also the x-terms, and the constant terms.

The solution of $ax^2+bx+c=0$ gives a formula, or shorthand law for writing the roots of any equation of that form.

Completing the square and solving,

$$\mathbf{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

This is the formula for writing the roots directly without completing the square. It is the final result that is always arrived at by completing the square, and it may always be written down at once.

Notice there are *two* roots, viz.:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}.$$

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

Solve the following by the formula:

1. $x^2-10x-24=0$

$$x=5\pm\sqrt{25+24}$$
$$x=12 \text{ and } -2$$

2. $2x^2-13x+15=0$

$$x=\frac{13}{4}\pm\sqrt{\frac{169}{16}-\frac{15}{2}}$$
$$x=\frac{13}{4}\pm\frac{7}{4}$$
$$x=5 \text{ and } 1\tfrac{1}{2}$$

By the use of this formula write by inspection the roots of the equations at the end of Exercise 171.

TO FIND APPROXIMATE VALUES OF ROOTS OF QUADRATIC EQUATIONS

395. Observe the following process for calculating approximate roots:

(1)

$$x^2-9x+16=0$$
$$x^2-9x+(\tfrac{9}{2})^2=\tfrac{81}{4}-16=\tfrac{17}{4}$$
$$x-\tfrac{9}{2}=\pm\tfrac{1}{2}\sqrt{17}$$
$$x=4\cdot5\pm2.062-$$
$$x=6.562- \text{ and } 2.438+$$

(2)

$$x^2-12x+25=0$$
$$x^2-12x+6^2=6^2-25=11$$
$$x-6=+\tfrac{1}{2}\sqrt{11}$$
$$x=6\pm1.658+$$
$$x=7.658+ \text{ and } 4.342-$$

Observe in each case whether the sum of the roots equals the coefficient of x with reversed sign.

Exercise 172

Find the approximate roots to two places of decimals, of the following:

1. $x^2-3x-8=0$
2. $x^2-5x+3=0$

Exercise 173

Solve the following equations:

1. $x^4+4x^2-45=0$
2. $x^{\frac{1}{2}}+x^{\frac{1}{4}}-30=0$
3. $x^6-5x^3-24=0$
4. $2\sqrt[3]{x}+3\sqrt[6]{x}=6$
5. $x+6\sqrt{x}-20=0$
6. $x^{\frac{2}{3}}+4x^{\frac{1}{3}}-5=0$
7. $x^4-5x^2-36=0$
8. $\sqrt{x}-3\sqrt[4]{x}=28$
9. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}-8=0$
10. $2x^3+5\sqrt{x^3}=7$
11. $x^4+4x^2-32=0$
12. $x^6+2x^3-80=0$
13. $x-5\sqrt{x}-14=0$
14. $4x^{\frac{1}{3}}+x^{\frac{1}{6}}-39=0$

397. Some expressions are in quadratic form *with reference to a compound expression*, such for example as,

$$(x+2)^2-(x+2)=12 \text{ and } x+3+2\sqrt{x+3}-3=0$$

These equations may be solved by factoring, the first one for $(x+2)$ and the second one for $\sqrt{x+3}$.

Exercise 174

Solve the following by factoring:

1. $x-8-\sqrt{x-8}=20$
2. $(x-2)^2-3(x-2)=10$
3. $x+6-2\sqrt{x+6}=8$
4. $(x^2-5)^2-4x^2+20=77$
5. $x+4+(x+4)^{\frac{1}{2}}=20$
6. $(x^2+8)^2-5x^2-40=84$

398. Some equations may be put in the quadratic form by *adding a number* to both members. For example,

$$x^2-4x+\sqrt{x^2-4x+12}=8$$

may be put in quadratic form by adding 12, thus:

$$x^2-4x+12+\sqrt{x^2-4x+12}=20$$

This is in the quadratic form with reference to $x^2-4x+12$.

By factoring, $\sqrt{x^2-4x+12}=-5 \text{ and } 4$

Squaring, $x^2-4x+12=25 \text{ and } 16.$

$$x^2-4x-13=0$$

$$\text{and } x^2-4x-4=0$$

The last two equations are ordinary quadratic equations.

Exercise 175

Solve the following equations in quadratic form:

1. $3x^4+5x^2-8=0$
2. $x^2-7x-\sqrt{x^2-7x+1}=5$
3. $x^{\frac{1}{2}}+5x^{\frac{1}{4}}+6=0$
4. $x^2-6x-\sqrt{4x^2-24x}=8$
5. $\sqrt{x}-3\sqrt[4]{x}=21$
6. $x^2+\sqrt{x^2-5x+3}=5x+3$
7. $x^3-5x^{\frac{3}{2}}-24=0$
8. $x^2-2x-\sqrt{9x^2-18x}=4$
9. $x^{-\frac{4}{3}}-5x^{-\frac{2}{3}}+4=0$
10. $(x^2+3)^2-5(x^2+3)=14$
11. $2x^{-\frac{2}{3}}-x^{-\frac{1}{3}}-45=0$
12. $x^2-6x-3=2\sqrt{x^2-6x}$
13. $(x-1)^{\frac{4}{3}}+(x-1)^{\frac{2}{3}}=2$
14. $x^2-5x+\sqrt{4x^2-20x}=48$
15. $(x-5)^2-x+5=110$
16. $\sqrt{2x^2+14x+2}=x^2+7x-3$

GRAPHICAL SOLUTION OF QUADRATICS

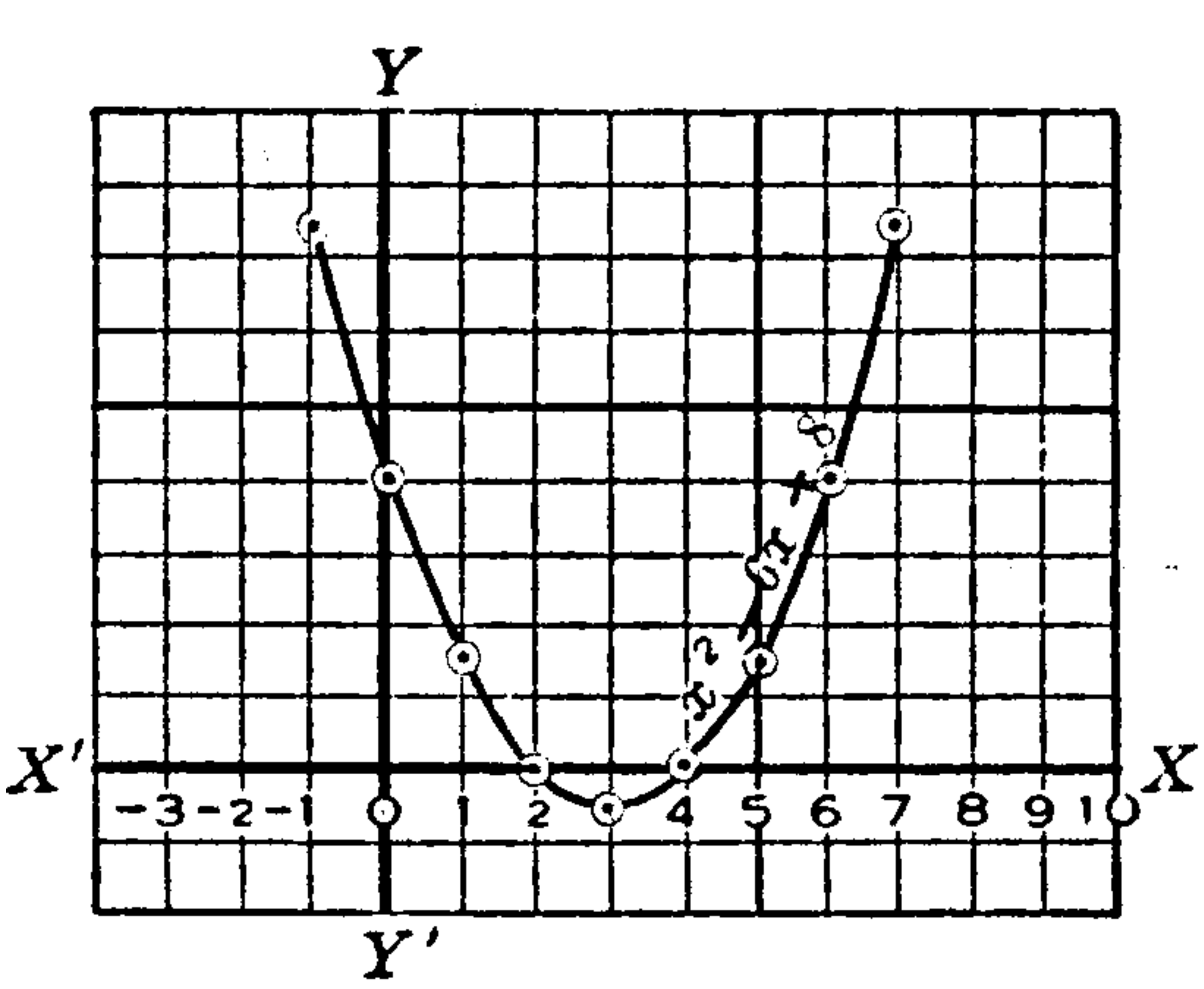
399. The graphical solution of quadratic equations makes the meaning of the roots, and the possibility of solutions, somewhat clearer.

To solve graphically the equation

$$x^2-6x+8=0.$$

First graph the function x^2-6x+8 for the values:

$$x=-1,\quad 0,\quad 1,\ 2,\quad 3,\ 4,\quad 5,\quad 6,\quad 7, \text{ etc.,}$$
$$x^2-6x+8=+15,\ +8,\ +3,\ 0,\ -1,\ 0,\ +3,\ +8,\ +15, \text{ etc.,}$$



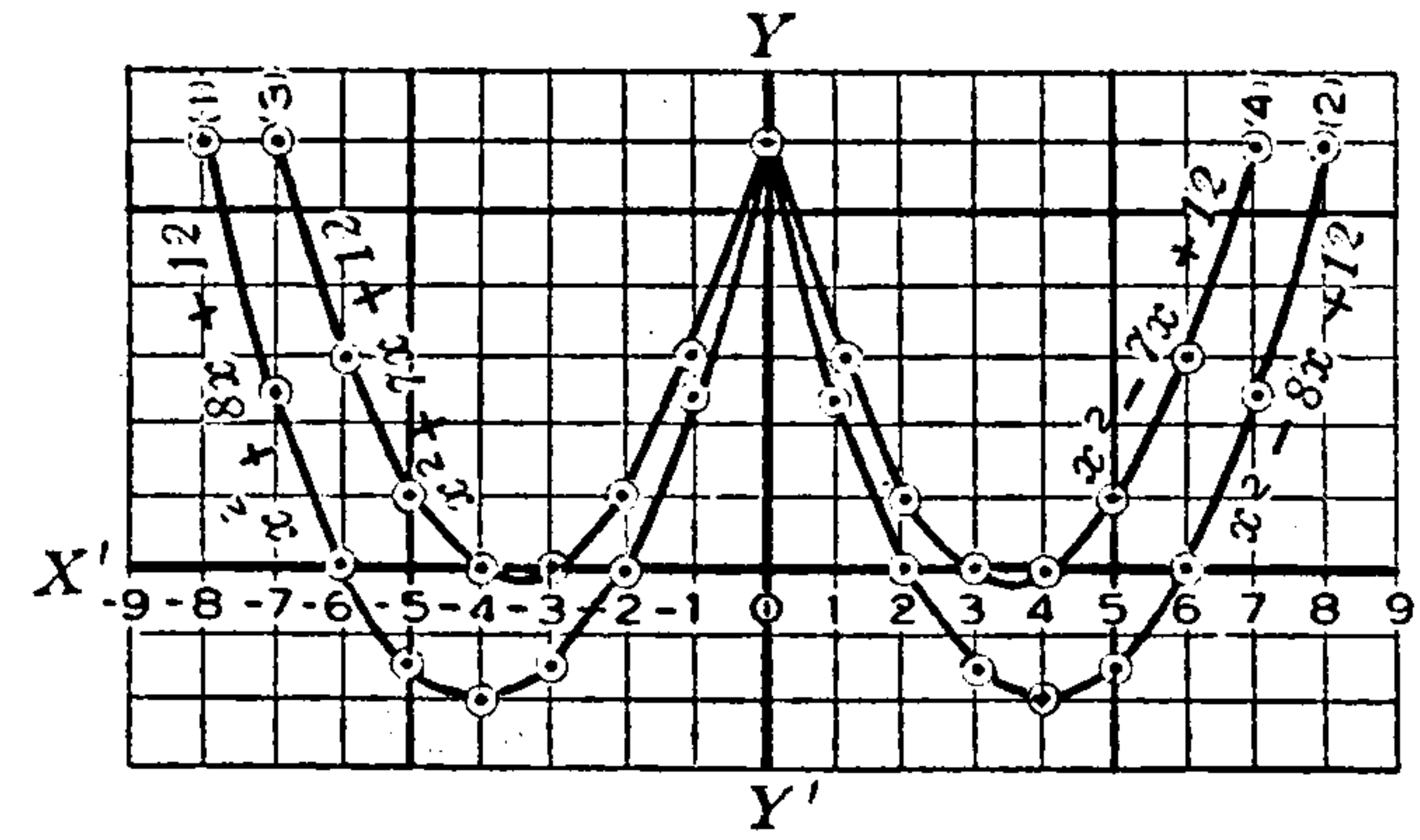
Scale
1 = 1 horizontal space
2 = 1 vertical space

Graph of x^2-6x+8

Plotting these points and connecting them as in the figure we have the graph of x^2-6x+8. To ask for the values of x that give $x^2-6x+8=0$, is to ask what are the x-values of the crossing-points of the graph over the horizontal. Clearly these values are $x=+2$ and $x=+4$.

The curve of the figure is called a **parabola** and any quadratic like x^2+px+q always gives a *parabola* for its graph.

400. This figure gives the graphs of four quadratics obtained by keeping the constant term equal to $+12$ and changing the coefficient of the x-term only.



Graph of $x' + ax + 12$
for $a = +8$ (1)
$a = -8$ (2)
$a = +7$ (3)
$a = -7$ (4)
Scale
2 = 1 *vertical* space

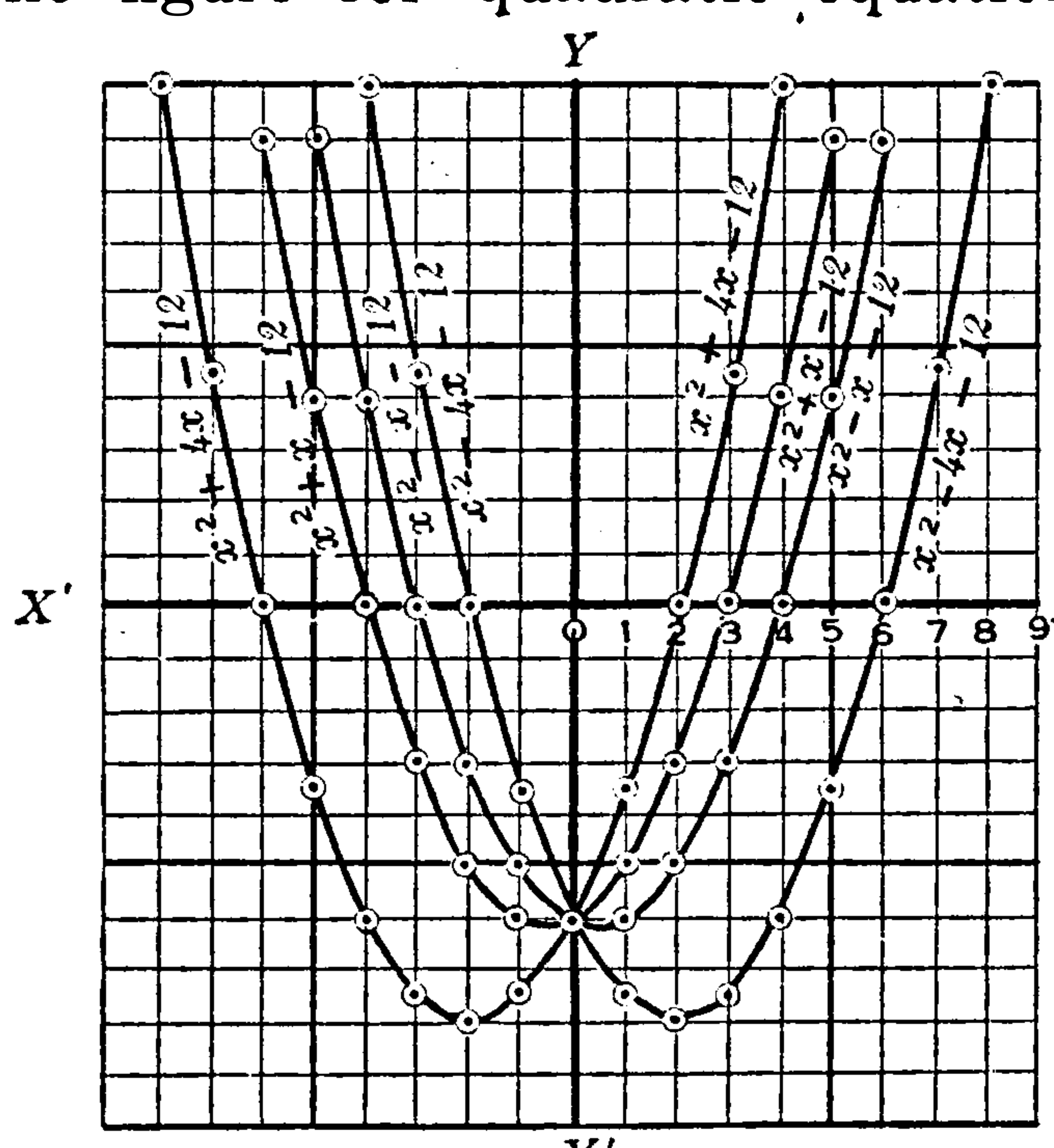
The quadratics $x^2+7x+12$ and $x^2-7x+12$ give the same shape of curve; either being turned over the vertical axis gives the other. The same is true of the graphs of $x^2+8x+12$ and $x^2-8x+12$.

This may be expressed by saying that reversing the sign of the coefficient of x in the quadratic, turns the graph over around the vertical axis.

The roots of such pairs of quadratics are numerically equal, but of opposite signs.

Give the roots from the figure for quadratic equations made by putting each of the four quadratic trinomials equal to 0.

All four of the graphs go through the point $+12$ on the vertical.



Graph of $x^2 + ax - 12$
for $a = +4$
$a = -4$
$a = +1$
$a = -1$
Scale
2 = 1 *vertical* space

401. This figure shows the graphs of quadratics all of which have the constant term -12. Compare the graphs of the pairs:

$$\begin{cases} x^2+x-12 \\ x^2-x-12, \end{cases}$$

$$\begin{cases} x^2+4x-12 \\ x^2-4x-12. \end{cases}$$

Exercise 176

Solve the following quadratic equations graphically:

1. $x^2-3x-10=0$
2. $x^2+3x-10=0$
3. $x^2-5x-6=0$
4. $x^2+x-20=0$
5. $x^2-x-20=0$
6. $x^2+5x=0$

CHARACTER OF THE ROOTS OF QUADRATIC EQUATIONS

403. The character of the roots of any complete quadratic equation is determined by examining the solutions of:

$$\mathbf{ax^2+bx+c=0}$$

In this discussion it is assumed that a, b, and c are real numbers, a is greater than zero, and b and c are either positive or negative.

Denoting the roots by r_1 and r_2, we have the values:

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The nature of the two roots, as real or imaginary, rational or irrational, depends on the value of b^2-4ac.

The expression b^2-4ac is called the **discriminant** of the roots.

404. Observing the formulas for r_1 and r_2, it is evident that:

1. *When the discriminant is a* ***square*** *the roots are real, rational, and unequal.*

2. *When the discriminant is equal to* **zero** *the roots are real, rational, and equal.*

3. *When the discriminant is a* ***positive number not a square*** *the roots are real and conjugate surds.*

4. *When the discriminant is a* ***negative number*** *the roots are conjugate complex numbers.*

A *complex number* is a number of the form $a+b\sqrt{-1}$, a and b denoting real numbers.

The numbers $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$, are *conjugate complex numbers.*

405. It follows that we can determine the nature of the roots of any quadratic equation without solving it. For example:

$$3x^2-7x+2=0$$

In this equation $b^2-4ac=25$. Since 25 is a square, the roots are real, rational, and unequal. But take the equation

$$5x^2-4x+2=0$$

In this equation $b^2-4ac=-24$. Since -24 is a negative number, the roots are *conjugate complex numbers.*

Exercise 177

By the use of the discriminant determine the nature of the roots of each of the following equations:

1. $4x^2-7x+3=0$

2. $x^2-7x-8=0$

3. $2x^2-4x+2=0$

4. $x^2+5x+6=0$

5. $5x^2+8x-2=0$

6. $x^2-3x+5=0$

7. $7x^2-5x+1=0$

8. $x^2+3x+5=0$

9. $4x^2-4x+1=0$

10. $x^2-5x-9=0$

11. $4x^2+6x-4=0$

12. $x^2-5x+8=0$

13. For what values of n will $2x^2+nx+8=0$ have equal roots? Irrational roots?

14. For what value of a will $ax^2-12x+6=0$ have equal roots? Imaginary roots?

15. For what values of n will $3x^2+2nx+3=0$ have equal roots? Imaginary roots?

16. For what values of c will $5x^2-10x+c=0$ have equal roots? Real roots? Imaginary roots?

17. For what values of n will $9x^2+nx+x+1=0$ have equal roots? Find the corresponding values of x.

406. By dividing both members of the general quadratic equation, $ax^2+bx+c=0$ by the coefficient of x^2, the equation becomes of the form:

$$\mathbf{x^2+2px+q=0}$$

in which p and q are positive or negative, integral or fractional, and $2p$ is any coefficient of x.

The solutions of this equation are, by § 394 or § 403.

$$r_1=-p+\sqrt{p^2-q}$$
$$r_2=-p-\sqrt{p^2-q}$$

The sum of the two roots of $\boldsymbol{x^2+2px+q=0}$ *is* $-2p$, *the coefficient of* x *with reversed sign.*

The product of the two roots of $\boldsymbol{x^2+2px+q=0}$ *is* q, *the constant term of the equation.*

407. The two foregoing principles enable us to form quadratic equations with given roots.

If the roots of a quadratic equation are -9 and 5, the coefficient of x is 4, and the constant term is -45. The equation then is

$$x^2+4x-45=0$$

It has already been proved §§ 215, 384-5, that if $(x+9)(x-5)=0$, the roots are -9 and 5.

Observe that the known numbers in $(x+9)(x-5)=0$, are the roots of the equation *with their signs reversed.*

TO FORM A QUADRATIC EQUATION WITH GIVEN ROOTS

408. Rule.— *Subtract each of the roots from x and place the product of the two remainders equal to zero.*

The equation whose roots are 6 and -7 is

$$(x-6)(x+7)=0, \text{ or}$$
$$x^2+x-42=0.$$

Exercise 179

Factor the following:

1. a^2-4a+1	2. x^2-8x-2	3. $n^2-6n+11$
4. a^2+6a-3	5. x^2+4x-4	6. $n^2-6n+13$
7. a^2-2a+4	8. x^2+8x-8	9. $n^2+4n-16$
10. a^2+5a-1	11. x^2-3x+1	12. $n^2+9n+23$

PROBLEMS IN QUADRATIC EQUATIONS

410. Since quadratic equations have two roots, a problem whose solution involves such an equation apparently has two values of the unknown number, or two roots.

Both roots may satisfy the equation, but only one of them may satisfy the *conditions of the problem.* Especially is this true when the roots are surds or imaginary.

In solving problems that involve quadratics, we should examine the roots of the equation and reject any root that does not satisfy the requirements of the problem.

Exercise 180—Problems in Quadratics

Solve the following problems:

1. The sum of two numbers is 42, and their product is 416. Find the two numbers.

2. The sum of the squares of three consecutive numbers is 590. Find the three numbers.

3. A rectangular field of 4 acres is 12 rods longer than it is wide. What are the dimensions?

4. The quotient of one number divided by another is 7, and their product is 2800. Find the numbers.

5. If the sum of the squares of three consecutive even numbers is 980, what are the numbers?

6. What is the price of eggs per dozen when 5 less for 50¢ increases the price 6¢ a dozen?

7. Find two consecutive odd numbers the sum of whose squares exceeds 20 times the larger number by 94.

8. The perimeter of a rectangular field is 84 rods, and the area is 432 square rods. Find the dimensions.

9. The sum of two numbers is 24, and their product is 139. Find the numbers and prove your answer.

10. The difference between two numbers is 16, and their product is 1380. Find the numbers.

11. The perimeter of a rectangular field is 114 rods, and the area is 5 acres. Find the dimensions.

12. Solve the formula $d = \frac{1}{2}gt^2$ for t and g.

13. The sum of two even numbers is 48, and the sum of their squares is 1224. Find the numbers.

14. The sum of two numbers is 40, and their product is $398\frac{1}{4}$. Find the numbers. Prove your answer.

15. The sum of two numbers is 96, and their product is 18 times as much. Find the numbers.

16. Solve the formula $a^2 = b^2 + c^2$ for b and c.

17. The hypotenuse of a right triangle is 9 feet longer than one leg and 2 feet longer than the other leg. Find the three sides of the triangle.

18. At 15¢ a square foot, it cost $99 to lay a parquet floor in a room whose length is 8 feet more than its width. Find the dimensions of the floor.

19. The dimensions of a certain rectangle and its diagonal are represented by three consecutive even numbers. What are the dimensions of the rectangle?

20. A carpenter worked 30 days more than he received dollars per day for his labor and earned $175. How many days did he work and how much did he receive per day?

21. Two numbers differ by 1. The square of their sum exceeds the sum of their squares by 220. Find the numbers.

22. There are 32 sq. yd. in a rectangle whose length is 18 times the width. Find the length in feet.

23. Find two numbers whose difference is 6, and whose sum multiplied by the smaller number is 756.

24. Find the side of a square whose area is doubled by increasing its length 9 yd. and its width 6 yd.

25. One square field is 10 rd. longer than another, and the area of both is 1108 sq. rd. Find the length of each.

26. Find the numbers the sum of whose two digits is 13 and the sum of the squares of whose digits is 89.

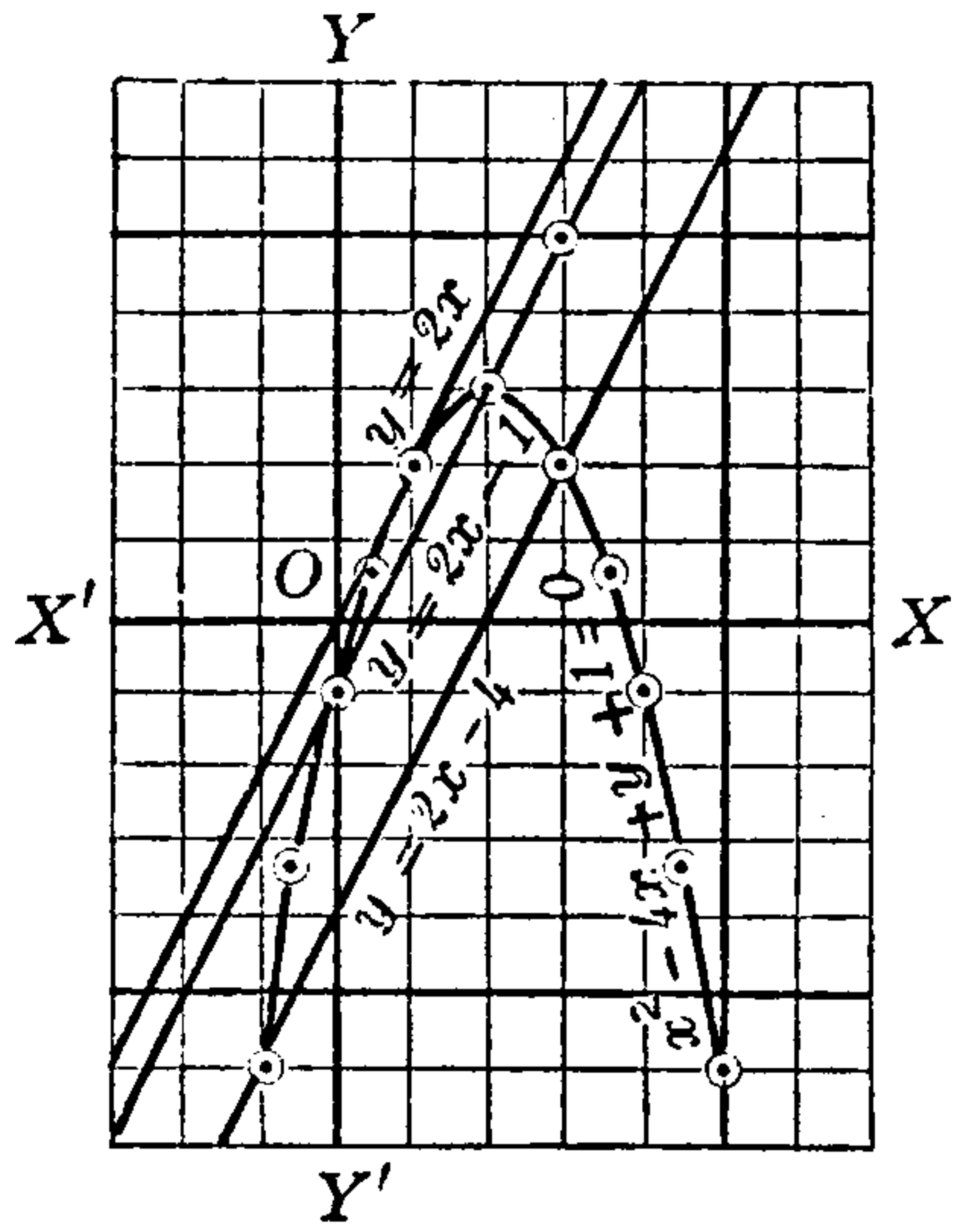
27. The number of square inches in the surface of a cube exceeds the number of inches in the sum of its edges by 1170. Find the volume of the cube.

28. A man bought a piece of land for $4050. He sold it at $53 an acre, making a profit equal to the cost of 16 acres. How many acres did he buy?

29. A merchant sold some damaged goods for $24 and lost a per cent equal to the number of dollars he paid for the goods. Find the cost of the goods.

30. The length of a rectangle exceeds its width by 7 rd. If the dimensions were increased 5 rd., it would contain 5 acres. Find the dimensions of the rectangle.

31. A merchant bought lace for $100. He kept 30 yards and sold the remainder for as much as it all cost, gaining 75¢ a yard. How many yards did he buy?



Scale
1 = 1 horizontal space
1 = 1 vertical space

On the same reference lines, graphing (2) for

$$\begin{cases} x=+2, \ -2 \\ y=+3, \ -5 \end{cases}$$

gives the straight line marked $y=2x-1$ in the figure.

The solutions sought are the x- and y-distances of the crossing-points of the graphs of (1) and (2).

The x- and y-values must be so paired that both numbers of each pair belong to the same crossing-point.

The solutions are: $x=0$, $y=-1$, and $x=+2$, $y=+3$.

414. The graph of (1) is a **parabola** and any two-letter equation of the second degree with only *one variable raised to the second power* and without an xy-term, gives a **parabola** for its graph.

415. Suppose a line to start from the position marked $y=2x-1$, moving across the parabola *parallel* to the starting position to the line $y=2x$. In every position there would be *two* crossing-points until the position $y=2x$ is reached. At this position the two crossing-points blend into *one*, the line becoming *tangent* to the parabola.

Beyond the position $y=2x$ there would be no crossing-point of the line and the parabola.

Starting from the line $y=2x-1$ and moving parallel to itself toward the right, there would always be *two* crossing-points. Recalling that every crossing-point gives a value of x and of y, we observe that:

I. There are in general ***two solutions*** *of a system made up of a parabolic and a linear equation.*

II. When the line is tangent to the parabola there is but ***one solution,*** *or since the two crossing-points coalesce, we may say* ***two equal*** *solutions.*

III. For an equation representing a line **beyond** *the tangent position there is* **no real solution.** *Algebra shows that there are two solutions even here, but that they are* **imaginary.**

416. The solution just given is the **graphical** solution of the system. We now give the **algebraic** solution of the same system.

Writing the equations thus:

$$\begin{cases} x^2-4x+y+1=0 & (1) \\ y=2x-1 & (2) \end{cases}$$

substitute the value of y from (2) in (1), simplify, and find:

$$x^2-2x=0$$

Whence, $x=0$, and $+2$

Substituting these values of x in (2), we find:

$$y=-1, \text{ and } +3$$

The solutions are the number pairs:

$$x=0,\ x=+2,$$
$$\text{and} \quad y=-1,\ y=+3$$

These values agree with those of the graphical solution.

Exercise 181

Solve the following systems algebraically:

1. $\begin{cases} x^2+3x-y=18 \\ y-2=2x \end{cases}$

2. $\begin{cases} 2x^2-6x+y=8 \\ y-4x=-4 \end{cases}$

3. $\begin{cases} y^2-2y+x=5 \\ x-2y=3 \end{cases}$

4. $\begin{cases} y^2-5y+3x=6 \\ 2y-3x=4 \end{cases}$

5. $\begin{cases} x^2-y=5 \\ 3x-y=-5 \end{cases}$

6. $\begin{cases} 3x^2-9x-y=2 \\ 3x-y=2 \end{cases}$

417. Solve next the system $x^2+y^2=25 \quad y=\pm\sqrt{25-x^2}$(1)

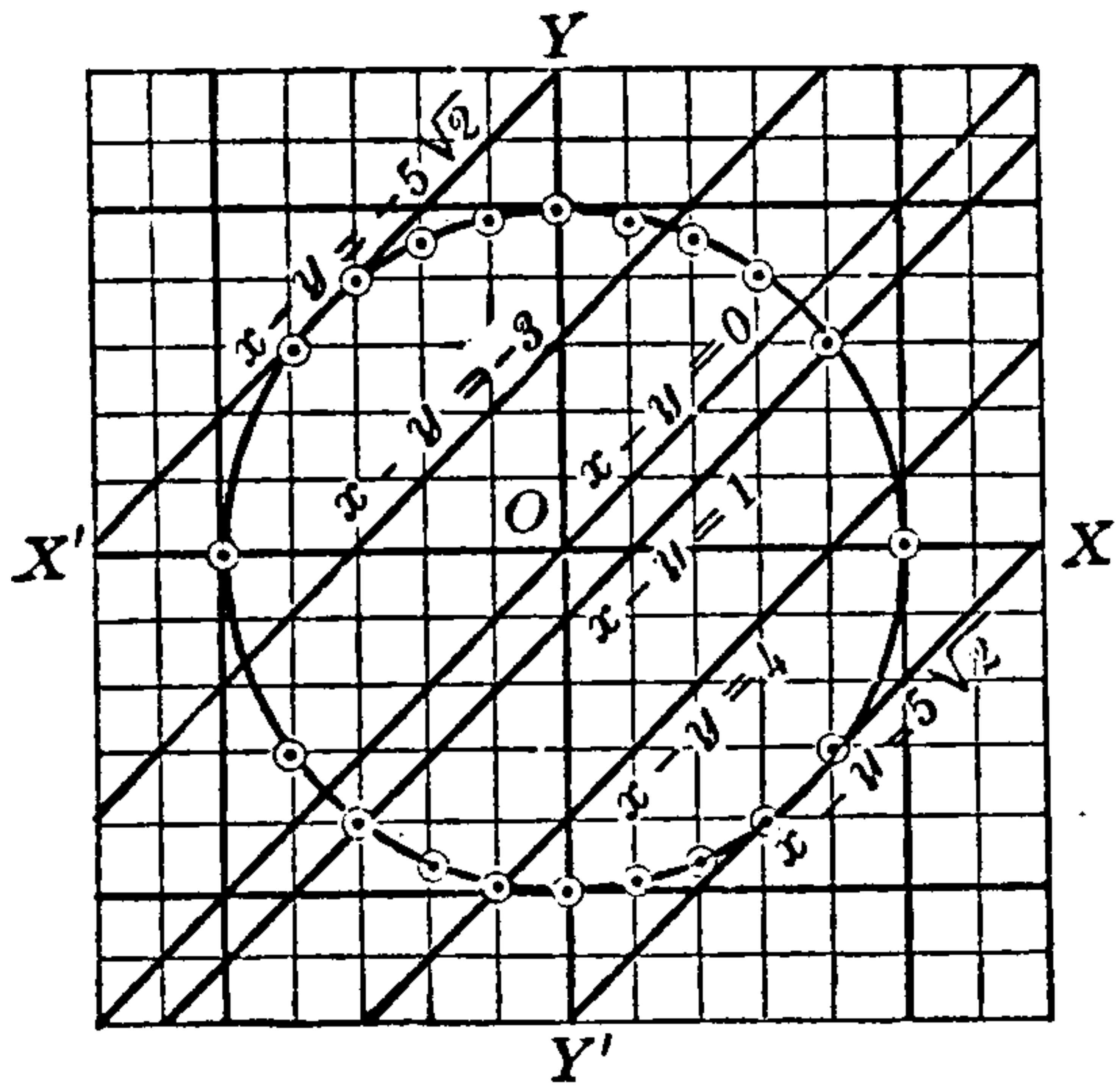
or

$$\begin{cases} x^2+y^2=25, \text{ or } y=\pm\sqrt{25-x^2} & (1) \\ x-y=1, \text{ or } y=x-1 & (2) \end{cases}$$

Graphing (1) $y=\pm\sqrt{25-x^2}$ using

$x=$ +6, +5, +4, +3, +2, +1, 0, −1, −2, −3, −4, −5, −6, etc., and calculating y from $y=\pm\sqrt{25-x^2}$, find

$y=$*imag.* 0, ±3, ±4, ±4.6, ±4.9, ±5, ±4.9, ± 4.6, ±4, ±3, 0, *imag.*, etc.



Scale
1 = horizontal space
1 = vertical space

Graphing these pairs, laying off the values with double sign *both upward and downward*, obtain the circle of the figure.

Graphing now the line $y=x-1$, obtain the straight line of the figure.

The crossing-points give the following solutions:

$$x=+4,\ x=-3,$$
$$y=+3,\ y=-4.$$

This is the *graphical solution.*

Suppose a line should start from the position $x-y=1$ and move upward across the circle, keeping parallel to $x-y=1$, through the positions $x-y=0$, $x-y=-3$, to $x-y=-5\sqrt{2}$, or downward through the position, $x-y=4$ to $x-y=5\sqrt{2}$. In every position the line gives two crossing-points with the circle, until the *tangent* positions are reached, where the two crossing-points become *one point* of contact.

For a line *beyond* the tangent positions the system would give two *imaginary* solutions. For the tangent positions of the line we might again say there are *two equal* solutions.

For the upper tangent-point $x=-\frac{5}{2}\sqrt{2}$, $y=+\frac{5}{2}\sqrt{2}$ and for the lower tangent-point, $x=+\frac{5}{2}\sqrt{2}$, $y=-\frac{5}{2}\sqrt{2}$.

420. The *algebraic solution* is obtained by substituting the value of y from (1) in (2), obtaining

$$16x^2+25[\tfrac{4}{5}(x-5)]^2=400$$

Reducing, $\quad 32x^2-160x+400=400$

Or, $\quad x^2-5x=0$

Whence, $x=0$, and $+5$, and from (1) $y=-4$, and 0.

The graph shows that the 0-value of x must be paired with the -4-value of y, and that the $+5$ and 0 also belong together.

The graph for the equation $16x^2+25y^2=400$, is an **ellipse.**

Moving this line $4x-5y=20$ parallel to itself across the ellipse shows there are always *two* crossing-points, and hence *two pairs of values of x and y*, save for the *tangent positions*, where there would be *only one* pair or, as we prefer to say, *two equal* pairs.

An algebraic solution would show that when the line does not touch the ellipse there would be two *imaginary* values of x and y.

421. A quadratic equation with no xy-term but containing the square-terms of both variables, the coefficients of these terms being of the same sign, gives a graph that is an *ellipse.*

Exercise 183

Solve the following systems algebraically:

1. $\begin{cases} 2x-3y=0 \\ 4x^2+9y^2=36 \end{cases}$ 2. $\begin{cases} x-3y=2 \\ 4x^2+9y^2=36 \end{cases}$ 3. $\begin{cases} 3x-5y=8 \\ x^2+25y^2=25 \end{cases}$

4. $\begin{cases} 7x-4y=10 \\ x^2+16y^2=16 \end{cases}$ 5. $\begin{cases} 5x-3y=3 \\ 9x^2+y^2=9 \end{cases}$ 6. $\begin{cases} 10x-3y=5 \\ 49x^2+y^2=49 \end{cases}$

422. Solve the system:

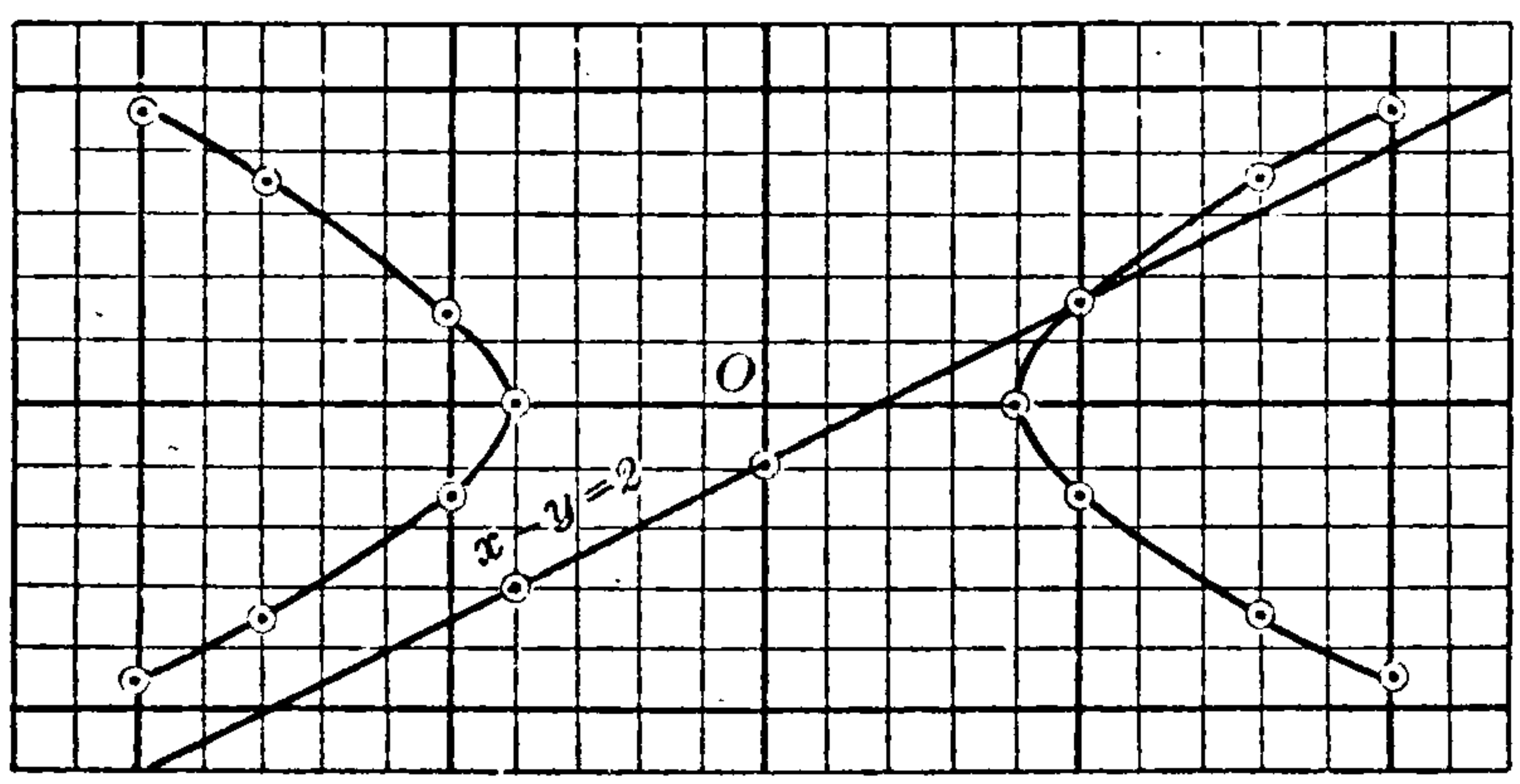
$$\begin{cases} x^2-y^2=16 & \text{(1), or } y=\pm\sqrt{x^2-16} \\ x-\ y=2 & \text{(2), or } y=x-2 \end{cases}$$

In equation (1) for all values of x between -4 and $+4$ the values of y are imaginary. Calculate y for the given x-values, find:

$x = +10,$	$+8,$	$+5,$	$+4,$	$-4,$	$-5,$	$-8,$	$-10,$	etc.
$y = \pm 9.2,$	$\pm 6.9,$	$\pm 3,$	$0,$	$0,$	$\pm 3,$	$\pm 6.9,$	$\pm 9.2,$	etc.

Plotting these points, drawing the graph, and graphing equation (2) for $x=0$, $y=-2$, and $x=-4$, $y=-6$, obtain the picture of the figure shown.

The graph of equation (1) is a **hyperbola.** It has two disconnected parts, or *branches.* There is but *one* crossing-



Scale
1 = 1 horizontal space
2 = 1 vertical space

point of the line and curve. The figure shows why. The graph shows the x- and y-values for this crossing-point to be $x = +5$, and $y = +3$.

423. The algebraic solution gives by substituting the value of y from (2) in (1) $x^2 - (x-2)^2 = 16$

Reducing, we find, $\qquad 4x = +20,$

or, $\qquad x = +5.$

This value of x, substituted in equation (2), gives

$$y = +3.$$

These values of x and y agree with the graphical solution.

424. A quadratic equation having both x^2- and y^2-terms with opposite signs, no xy-term being present, always gives a **hyperbola** for its graph.

Could the straight line be turned around so that it would cut *both branches* of the hyperbola? How many values of x and of y would there be?

Exercise 184

Solve the following systems algebraically:

1. $\begin{cases} x^2-y^2=7 \\ x-y=1 \end{cases}$ 2. $\begin{cases} x^2-y^2=13 \\ x-y=1 \end{cases}$ 3. $\begin{cases} x^2-y^2=45 \\ x-y=5 \end{cases}$

4. $\begin{cases} x^2-y^2=27 \\ x-y=3 \end{cases}$ 5. $\begin{cases} x^2-y^2=55 \\ x-y=5 \end{cases}$ 6. $\begin{cases} x^2-y^2=80 \\ x-3y=6 \end{cases}$

425. Solve the system:

$$\begin{cases} xy=12, \text{ or } y=\dfrac{12}{x} & (1) \\ y-x=1, \text{ or } y=x+1 & (2) \end{cases}$$

In equation (1) calculate y for the following assumed values of x,

$$\begin{array}{lrrrrrrrrrrrr} x= & +12, & +6, & +4, & +3, & +2, & +1, & -1, & -2, & -3, & -4, & -6, & -12 \\ y= & +1, & +2, & +3, & +4, & +6, & +12, & -12, & -6, & -4, & -3, & -2, & -1 \end{array}$$

Plot these points, and draw the graph, obtaining a curve for $xy=12$. Show both branches of the curve.

Both branches together are spoken of as a single curve, the *hyperbola.*

Graphing equation (2) on the same axes, using the following points,

$$\begin{array}{lrrrr} x= & +3, & 0, & -1, & -4 \\ y= & +4, & +1, & 0, & -3, \end{array}$$

the straight line graph for $y=x+1$ is obtained.

Exercise 185

1. $\begin{cases} xy=12 \\ y-x=4 \end{cases}$

2. $\begin{cases} xy=36 \\ x-y=-5 \end{cases}$

3. $\begin{cases} xy=20 \\ x-4y=2 \end{cases}$

4. $\begin{cases} 3xy=21 \\ x-8y=-1 \end{cases}$

5. $\begin{cases} 5xy=150 \\ x-y=-1 \end{cases}$

6. $\begin{cases} 7xy=98 \\ x-5y=-3 \end{cases}$

429. The main use of the graphical solution of equations to pupils is to enable them to see the meaning of solutions, and to understand why roots are paired in a certain way.

For practical work of solving equations the algebraic solution, as given in §§ 416, 418, 420, 423 and 428, should always be used. In the exercises that follow the algebraic method is to be employed.

Exercise 186

Solve the following systems and pair the roots properly:

1. $\begin{cases} x^2+y=27 \\ x-y=3 \end{cases}$

2. $\begin{cases} x-y^2=4 \\ x+y=10 \end{cases}$

3. $\begin{cases} y-x^2+x=1 \\ x=y-4 \end{cases}$

4. $\begin{cases} x^2+y^2=20 \\ x+y=6 \end{cases}$

5. $\begin{cases} x^2+y^2=26 \\ x-y=6 \end{cases}$

6. $\begin{cases} x^2+y^2=73 \\ y-2x=13 \end{cases}$

7. $\begin{cases} 3x^2+8y^2=147 \\ x-y=2 \end{cases}$

8. $\begin{cases} x^2+2y^2=89 \\ x+y=11 \end{cases}$

9. $\begin{cases} 5x^2+y^2=45 \\ x+2y=12 \end{cases}$

10. $\begin{cases} xy=10 \\ y-x=3 \end{cases}$

11. $\begin{cases} xy=18 \\ x+y=9 \end{cases}$

12. $\begin{cases} xy=24 \\ x-y=2 \end{cases}$

13. $\begin{cases} x+2y^2=10 \\ x-y=7 \end{cases}$

14. $\begin{cases} 3x^2-y=7 \\ y-5x=5 \end{cases}$

15. $\begin{cases} x^2=12-y \\ x-y=0 \end{cases}$

16. $\begin{cases} m+n=5 \\ mn=-14 \end{cases}$

17. $\begin{cases} m-n=3 \\ mn=18 \end{cases}$

18. $\begin{cases} x^2+xy+y^2=61 \\ x+y=9 \end{cases}$

19. $\begin{cases} a^2-ac+c^2=76 \\ a+c=14 \end{cases}$

20. $\begin{cases} x^2+y^2=65 \\ y=11-x \end{cases}$

21. $\begin{cases} m^2-mn+n^2=19 \\ m-n=3 \end{cases}$

22. $\begin{cases} 4x^2-3xy=91 \\ 3y-2x=1 \end{cases}$

23. $\begin{cases} m^2+n^2+mn=39 \\ m-n=3 \end{cases}$

24. $\begin{cases} xy+y^2=40 \\ y-3x=-4 \end{cases}$

25. $\begin{cases} 2a-5y=0 \\ a^2-3y^2=13 \end{cases}$

26. $\begin{cases} m+2x=27 \\ mx=85 \end{cases}$

27. $\begin{cases} a^2+b^2-a-b=18 \\ a+b=-5 \end{cases}$

28. $\begin{cases} x^2-y^2=65 \\ y-2x=-14 \end{cases}$

29. $\begin{cases} ab+a^2=40 \\ b-3a=-4 \end{cases}$

30. $\begin{cases} x^2-y^2=\frac{15}{4} \\ x-y=\frac{3}{2} \end{cases}$

31. $\begin{cases} c^2-5d^2=76 \\ 4c-5d=29 \end{cases}$

32. $\begin{cases} 2m-3n=9 \\ mn=6 \end{cases}$

33. $\begin{cases} 3xy+x^2-2y^2=52 \\ 2x+3y=36 \end{cases}$

34. $\begin{cases} 3m-2n=23 \\ m^2-2mn=45 \end{cases}$

35. $\begin{cases} 2c-d=35 \\ cd=57 \end{cases}$

36. $\begin{cases} y^2-yz=80 \\ y-z=16 \end{cases}$

37. $\begin{cases} m^2-n^2=16 \\ n+2m=13 \end{cases}$

38. $\begin{cases} xy=4 \\ x-y=0 \end{cases}$

39. $\begin{cases} 4m^2-9n^2=19 \\ 3n+2m=19 \end{cases}$

40. $\begin{cases} p^2+4q=76 \\ 3p-q=21 \end{cases}$

41. $\begin{cases} m^2+n^2-m-n=50 \\ mn=30 \end{cases}$

42. $\begin{cases} 3x^2-y^2=275 \\ x-3y=-5 \end{cases}$

430. Special Methods. Some systems may be conveniently solved by special methods as well as by substitution.

431. One of these special methods is to divide the given equations, member by member, obtaining a derived equation which, with one of the given equations, furnishes a system of equations equivalent to the given system, and then to solve the derived system.

(*a*) Observe carefully the following solution of the system:

$$\begin{cases} x^2-y^2=33 & (1) \\ x+y=11 & (2) \end{cases}$$

Dividing (1) by (2), $\quad x-y=3 \qquad (3)$

The system consisting of (2) and (3) is simpler than the given system and the simpler system gives $x=7$ and $y=4$.

These are all the roots, for (1) represents a hyperbola and (2) a straight line, and they cross in only one point.

(*b*) Solve the system:

$$\begin{cases} 36m^2-p^2=819 & (1) \\ 6m-p=-39 & (2) \end{cases}$$

Dividing (1) by (2), $\quad 6m+p=-21 \qquad (3)$

The system (2) and (3) is equivalent to the given system and its roots are:

$$m=-5 \text{ and } p=+9$$

Exercise 187

Solve the following systems, first dividing when possible and pairing results properly:

1. $\begin{cases} 9x^2-4y^2=308 \\ 3x-2y=14 \end{cases}$

2. $\begin{cases} m^2-n^2=64 \\ m+n=16 \end{cases}$

3. $\begin{cases} 9p^2-\frac{1}{q}=80 \\ 3p-\frac{1}{q}=8 \end{cases}$

4. $\begin{cases} \frac{m}{a^2}-\frac{m}{b^2}=mn \\ \frac{m}{a}-\frac{m}{b}=n \end{cases}$

The given system is then equivalent to the two systems:

$$\begin{cases} x^2 - y^2 = -5 \\ x + y = +5 \end{cases} \quad \text{and} \quad \begin{cases} x^2 - y^2 = -5 \\ x + y = -5 \end{cases}$$

Dividing the first equations by the second, obtain:

$$x - y = -1 \text{ and } x - y = +1$$

Combining these with the second equations of the derived system, we have:

$$x = 2 \text{ and } y = 3, \text{ and } x = -2 \text{ and } y = -3$$

(*c*) Solve the system:

$$x^2 + y^2 = 40 \qquad (1)$$

$$xy = 12 \qquad (2)$$

Multiplying (2) by 2 and adding to (1), obtain:

$$x^2 + 2xy + y^2 = 64$$

Or, $$x + y = \pm 8 \qquad (3)$$

Subtracting $2xy = 24$ from (1)

$$x^2 - 2xy + y^2 = 16,$$

Or, $$x - y = \pm 4 \qquad (4)$$

Now from (3) and (4) we form the four systems which are together equivalent to the given system, viz.:

$$\text{I.} \begin{cases} x + y = +8 \\ x - y = +4 \end{cases} \qquad \text{II.} \begin{cases} x + y = +8 \\ x - y = -4 \end{cases}$$

$$\text{III.} \begin{cases} x + y = -8 \\ x - y = +4 \end{cases} \qquad \text{IV.} \begin{cases} x + y = -8 \\ x - y = -4 \end{cases}$$

System I gives $x = 6$, $y = 2$, II, gives $x = 2$, $y = 6$, III, gives $x = -2$, $y = -6$, and IV gives $x = -6$, $y = -2$. Hence, the solutions of the given system are:

$$x = +6,\ +2,\ -2, \text{ and } -6,$$

$$y = +2,\ +6,\ -6, \text{ and } -2.$$

The system (1) and (2) are both quadratic equations, so that this problem lies a little beyond the limits set for this book. But the method in most of its parts is so like that for systems made up of one quadratic and one linear as to bring it within the pupil's comprehension. The reason there are so many solutions lies in the fact that the graph of (1) is a circle and of (2) a hyperbola, since a circle and a hyperbola, in general, cross each other in *four* points.

433. In the following list of exercises we shall include a few systems in two quadratics of the type of the last.

Exercise 188

Solve the following systems of equations:

1. $\begin{cases} x^2+y^2=10 \\ 6xy=18 \end{cases}$ **2.** $\begin{cases} 4r^2-6rs+9s^2=73 \\ rs=12 \end{cases}$

3. $\begin{cases} 64m^2+n^2=356 \\ 8mn=160 \end{cases}$ **4.** $\begin{cases} x^2+xy+y^2=49 \\ \dfrac{xy}{25}=3 \end{cases}$

5. $\begin{cases} x^2-xy=22 \\ xy-y^2=18 \end{cases}$ **6.** $\begin{cases} x^2+4xy+36y^2=224 \\ 12xy=96 \end{cases}$

Exercise 189

1. The sum of two numbers is 7 (or a), and the sum of their squares is 21 (or b). What are the numbers?

2. Find two numbers the difference of whose squares is 33 (or m), and the product of whose squares is 784 (or n).

3. The combined area of two square fields is $8\frac{1}{8}$ acres, and the sum of their perimeters is 200 rods. What is the area of each field?

4. The sum of the squares of two numbers is 91 (or p), and the difference of the numbers is 5 (or q). Find the numbers.

5. The difference of two numbers is 28, and half their product is equal to the cube of the smaller number. What are the numbers?

6. The area of the ceiling of a hall is 700 square feet, and its length is six feet less than four times the width. Find the dimensions.

7. The sum of two numbers is 13 (or s), and their product is 210 (or p). Find the numbers.

8. If the dimensions of a rectangle were each increased 1 foot, the area would be 99 square feet; if they were each diminished 1 foot, the area would be 63 square feet. What are the dimensions?

9. A number is expressed by two figures the sum of which is 14, and the sum of the squares of the digits exceeds the number by 11. Find the number.

10. The combined area of two adjoining square fields is 900 square rods, and it requires 150 rods of fence to inclose them. If they are so situated as to require the least amount of fence, what is the dimension of each?

11. The area of a rectangle is 192 square inches, and its diagonal is 20 inches. Find the dimensions.

12. A rectangular field contains 270 square rods. If it were two rods longer and one rod wider, it would contain 50 square rods more. Find the dimensions of the field.

13. A farmer bought 12 sheep and 4 calves for \$60. At the same prices, he could buy 3 more sheep for \$24 than calves for \$30. Find the price of each.

14. The perimeter of a rectangular piece of ground is 200 rods, and its area is 15 acres. Find the dimensions of the field.

15. The hypotenuse of a right triangle is 30 feet, and its area is 216 square feet. Find the length of the other two sides.

SUMMARY OF DEFINITIONS FOR REFERENCE AND REVIEW

(Definitions without page numbers are on page last indicated.)

CHAPTER I

The **factors** of a number are its **makers** by multiplication. (Page 8.)

An **equation** is an **expression of equality** between two equal numbers. (Page 11.)

The **value** of any letter in a number expression is the number or numbers it represents. (Page 12.)

An **unknown number** is a letter whose value in an equation is to be found. (Page 13.)

Solving an equation is finding the value of the unknown number, or numbers in it.

An **axiom** is a statement so evidently true that it may be accepted without proof.

In problem-solving the **notation** is the representation in algebraic symbols of the unknown numbers of the problem. (Page 15.)

The **statement** is the expression of the conditions of the problem in one or more equations.

CHAPTER II

Directed numbers or signed numbers are numbers whose units are **positive** or **negative.** (Page 21.)

The **absolute value** of a number is the number of units in it, regardless of sign. (Page 22.)

The + and − signs may denote either **operations** or **opposing** qualities of numbers. (Page 23.)

Algebraic notation is a method of expressing numbers by figures and letters. (Page 24.)

An **algebraic expression** is the representation of any number in algebraic notation.

A **term** is a number expression whose parts are not separated by the + or − sign.

A **monomial** is an expression of *one* term. (Page 25.)

A **polynomial** is an expression of *two* or *more* terms.

A **binomial** is a polynomial of *two* terms.

A **trinomial** is a polynomial of *three* terms.

A **coefficient** of a term is any factor of the term which shows how many times the other factor is taken as an addend.

Similar terms are terms which do not differ, or which differ only in their numerical factors.

Dissimilar terms are terms that are not similar.

Partly similar terms are terms that have a common factor.

The **value of an algebraic expression** is the number it represents when some particular value is assigned to each letter in the expression. (Page 26.)

CHAPTER III

Addition is the process of uniting two or more numbers into one number. (Page 27.)

The **addends** are the numbers to be added.

The **sum** is the number obtained by addition.

The **fundamental laws of addition** are the *law of order*, (the commutative law), and the *law of grouping* (the associative law). (Page 29.)

The **law of order** states that numbers may be added in *any* order.

The **law of grouping** states that addends may be grouped in *any* way.

CHAPTER IV

Subtraction is the process of finding one of two numbers when their sum and the other number are known. (Page 35.)

The **minuend** is the number that represents the sum.

The **subtrahend** is the given addend.

The **difference** or **remainder** is the number which added to the subtrahend gives the minuend.

The **symbols of aggregation** are the *parenthesis* ( ), the *brace* { }, the *bracket* [ ], and the *vinculum* ———. (Page 42.)

CHAPTER V

Algebraic number and **function** have the same meaning. (Page 50.)

The **independent number** is the number on which the function depends.

A **function** is a number that depends on some other number for its value. (Page 51.)

An **algebraic function** is a number whose dependence on another number is expressed in algebraic symbols.

CHAPTER VI

Equations are of two kinds, *identities* and *conditional equations.* (Page 60.)

An **identity** is an equation with *like* members, or members which may be reduced to the same form.

Substitution is putting a number symbol into a number expression in place of another which has the same value.

An **equation is satisfied** by any number which, when substituted for the unknown number, reduces the equation to an identity.

A **conditional equation** is an equation that can be satisfied by only *one* or by a *definite number* of values of the letters in it. (Page 61.)

A **root** of an equation is any value of the unknown number that satisfies the equation.

Transposition is the process of changing a term from one member of an equation to the other, by adding or subtracting the same number in both members. (Page 62.)

CHAPTER VII

Graphing means representing number-pairs, related sets of numbers, and number laws by pictures and diagrams. (Page 74.)

A **linear equation** is an equation in two unknowns both with exponent 1. (Page 81.)

The **graphical solution** of two linear equations is the point of intersection of the graphs of the equations. (Page 82.)

Simultaneous equations are equations that can be satisfied by the same values of x and y.

A **system** of equations is two or more equations considered together. (Pages 82 and 86.)

Non-simultaneous or **inconsistent** equations are equations which cannot be satisfied by any values of the unknowns. (Page 83.)

Dependent equations are equations in which one or more can be derived from another or others by some simple arithmetical operation. (Page 84.)

A **polynomial** is **arranged** when the exponents of some letter increase or decrease with each succeeding term. (Page 97.)

CHAPTER X

The **degree of a term** is indicated by the *sum* of the exponents of the literal factors. (Page 100.)

The **degree of an equation** in one unknown is the degree of the *highest power* of the unknown number.

A **simple equation,** or **linear equation,** is an equation which, when cleared and simplified, is of the *first* degree.

Checking or **verifying** a root of an equation is the process of proving that the root satisfies the equation.

CHAPTER XI

Division is the process of finding *one* of two numbers when their *product* and the *other* number are known. (Page 107.)

The **dividend** is the number to be divided and represents the product of the two numbers.

The **divisor** is the number by which we divide and represents one factor of the dividend.

The **quotient** is the result of division.

Any number with a zero-exponent equals 1. (Page 108.)

CHAPTER XIII

A **general number** is a letter or other number symbol that may represent *any* number. (Page 123.)

A **formula** is an expression of a *general principle,* or *rule,* in general number symbols and in the form of an equality. (Page 124.)

To **solve a formula completely** is to find the value of each general number in terms of the others. (Page 125.)

CHAPTER XIV

A **root** of a number is one of its equal factors. (Page 139.)

The **square root** of a number is one of the two equal factors whose product is the number. (Page 140.)

The **cube root** of a number is one of the three equal factors whose product is the number.

CHAPTER XVI

A **common divisor,** or **common factor,** of two or more numbers is an *exact* divisor of each of them. (Page 172.)

The **highest common factor** (h.c.f.) of two or more numbers is the product of all their common factors.

A **multiple** of a number is a number that is *exactly divisible* by it. (Page 175.)

A **common multiple** of two or more numbers is a number that is exactly divisible by each of them.

The **lowest common multiple** (l.c.m.) of two or more numbers is the product of all their *different* factors.

CHAPTER XVII

An **algebraic fraction** is the indicated division in fractional form of one number by another. (Page 179.)

The **numerator** is the number above the line.

The **denominator** is the number below the line.

The **terms** of a fraction are the numerator and denominator together.

An **integer,** or **integral number,** is a number no part of which is a fraction.

The **sign of a fraction** is the sign written before the line that separates the terms. (Page 180.)

Reduction of fractions is the process of changing their *form* without changing their *values*. (Page 181.)

A **mixed number** is a number one part of which is integral and the other part fractional. (Page 184.)

A **proper fraction** is a fraction which cannot be reduced to a whole or a mixed number.

An **improper fraction** is a fraction which can be reduced to a whole or a mixed number.

The **lowest common denominator** (l.c.d.) of two or more fractions is the l.c.m. of their denominators. (Page 187.)

The **reciprocal** of a fraction is the fraction *inverted*. (Page 193.)

CHAPTER XVIII

A **literal equation** is an equation in which there are two or more general numbers. (Page 198.)

A **general problem** is a problem all of the numbers in which are general numbers. (Page 207.)

CHAPTER XX

The **ratio** of one number to another is the quotient of the first number divided by the second. (Page 229.)

The **antecedent** is the first number of a ratio, and the **consequent** is the second number.

The **terms** of a ratio are the antecedent and consequent.

The **value of a ratio** is the quotient expressed in its lowest terms.

A **ratio of greater inequality** is a ratio in which the antecedent is greater than the consequent. (Page 231.)

A **ratio of less inequality** is a ratio in which the antecedent is less than the consequent.

A **proportion** is an equation of ratios. (Page 232.)

The **terms** of a proportion are the terms of the ratios.

The **extremes** of a proportion are the first and fourth terms; the **means** are the second and third terms.

A **mean proportional** is the second of three numbers which form a continued proportion, as x in $a:x \doteq x:b$. (Page 234.)

A **third proportional** is the third of three numbers that form a continued proportion.

A **fourth proportional** is the fourth of four numbers that form a proportion.

A **variable number,** or a **variable,** is a number which in a given problem, or discussion, may have different values. (Page 241.)

A **constant number,** or a **constant,** is a number that is not a variable.

One variable **varies as** another if, as they vary, their *ratio* remains *constant.*

CHAPTER XXI

Involution is the process of raising a number to a power whose exponent is a positive integer. (Page 244.)

The **exponent** of the power is the number which indicates how many times the number (the root or base) is taken as a factor.

The **base** of a power is the number which is raised to a power.

Evolution is the process of finding a root of a number. (Page 250.)

A **binomial quadratic surd** is a binomial surd whose surd term or terms are of the second order. (Page 276.)

Conjugate surds are two binomial quadratic surds that differ only in the sign of one of the terms.

An **irrational,** or **radical,** equation is an equation containing an *irrational root* of the unknown number. (Page 278.)

CHAPTER XXIII

A **quadratic equation** is an equation of the second degree in the unknown number. (Page 282.)

The **constant term** in a quadratic equation is the term that does not contain the unknown number.

A **pure quadratic equation** is an equation that does not contain the first power of the unknown number.

An **affected quadratic equation** is an equation that contains both the first and second powers of the unknown number.

Pure quadratics are often called **incomplete quadratics,** and affected quadratics are also often called **complete quadratics.**

The **discriminant** of the roots of $ax^2+bx+c=0$ is $b^2 - 4ac$. (Page 298.)

A **complex number** is a number of the form $a+b\sqrt{-1}$, a and b denoting *real* numbers.

Conjugate complex numbers are complex numbers which differ in the sign of the imaginary term.

CHAPTER XXIV

A **quadratic equation in two variables** is an equation in two variables, one or both of which are of the second degree. (Page 305.)

A **system of quadratic equations** is two or more quadratic equations considered together.

A **simultaneous system** is a system in which all the equations can be satisfied by the same values of the variables.

INDEX